

On Nash Dynamics of Matching Market Equilibria

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Abstract

In this paper, we study the Nash dynamics of strategic interplays of n buyers in a matching market setup by a seller, the market maker. Taking the standard market equilibrium approach, upon receiving submitted bid vectors from the buyers, the market maker will decide on a price vector to clear the market in such a way that each buyer is allocated an item for which he desires the most (a.k.a., a market equilibrium solution). While such equilibrium outcomes are not unique, the market maker chooses one (MAX-EQ) that optimizes its own objective — revenue maximization. The buyers in turn change bids to their best interests in order to obtain higher utilities in the next round's market equilibrium solution.

This is an $(n + 1)$ -person game where buyers place strategic bids to gain the most from the market maker's equilibrium mechanism. The incentives of buyers in deciding their bids and the market maker's choice of using the MAX-EQ mechanism create a wave of Nash dynamics involved in the market. We characterize Nash equilibria in the dynamics in terms of the relationship between MAX-EQ and MIN-EQ (i.e., minimum revenue equilibrium), and develop convergence results for Nash dynamics from the MAX-EQ policy to a MIN-EQ solution, resulting an outcome equivalent to the truthful VCG mechanism.

Our results imply revenue equivalence between MAX-EQ and MIN-EQ, and address the question that why short-term revenue maximization is a poor long run strategy, in a deterministic and dynamic setting.

1 Introduction

The Nash equilibrium paradigm in Economics has been based on a rationality assumption that each individual will maximize its own utility function. Making it further to a dynamic process where multiple agents play interactively in a repeated game, the Nash dynamics refers to a process where participants take turns to choose a strategy to maximize their own utility function. We pose the question of what impact of such strategic behavior of the seller, as the market maker, can have on Nash dynamics of the buyers in a matching market setting [28], where the seller has m products and there are n unit-demand potential buyers with different private values v_{ij} for different items.

We base our consideration on the market equilibrium framework: The market maker chooses a non-negative price vector and an allocation vector; the outcome is called a *market competitive equilibrium* if all items with positive prices are sold out and everyone gets his maximum utility at the corresponding allocation, measured by the difference between the buyer's value of the item and the charged price, i.e., $v_{ij} - p_j$. To achieve its own objective, i.e., revenue maximization, the market maker is naturally to set all items the highest possible prices so that there is still a supported allocation vector that clears the market and maximizes everyone's utility. This yields a seller-optimal outcome and is called a *maximum competitive equilibrium*. Alternatively, at another extreme, the outcome of the market can be buyers-optimal, i.e., a *minimum competitive equilibrium* where all items have the lowest possible prices to ensure market clearance and utility maximization. Maximum and minimum equilibria represent the contradictory interests of the two parties in a two-sided matching market at the two extremes; and, as shown by Shapley and Shubik [28], both equilibria always exist and their prices are unique.

Using market equilibrium as a mechanism for computing prices and allocations yields desirable properties of efficiency and fairness. The winners in the mechanism, however, will bid against the protocol to reduce their payments; further, the losers may increase their bids to be more competitive (to their best interests). The problem becomes an $(n + 1)$ -person game played in turn between the seller and the buyers where buyers make strategic moves to (mis)report their bids for the items, in response to other buyers' bids and the solution selected by the market maker.

It is well-known that the minimum equilibrium mechanism admits a truthful dominant strategy for every buyer [22, 8], resulting in a solution equivalent to the VCG protocol [30, 5, 15]. However, it is hardly convincing a rational economic agent who dominates the market to make a move that maximizes the collective returns of its buyers by sacrificing its own revenue. Clearly, the Nash equilibrium paradigm has always assumed that an economic agent is a utility maximizer; for the market maker, it implies revenue maximization. Further, success in bringing great revenue (in a short term) will bring a large number of eyeballs and lead to positive externalities, which could breed more success (in a long term). Indeed, quite a few successful designs, like the generalized second price auction for sponsored search markets [1] and the FCC spectrum auction for licenses of electromagnetic spectra [6, 23], do not admit a truthful dominant strategy.

Our interest is to find out the properties of the game between the seller and the buyers (among which they compete with each other for the items), especially with these two extremes of the solution concepts — their relationship proves to be significant in the incentive analysis of the buyers in the market. In particular, we are interested in the dynamics of the outcomes when the market maker adopts the maximum equilibrium mechanism and the buyers adopt a particular type of strategy, that of the best response strategy. With the best response strategy, a buyer would announce his bids for all items that would result in the maximum utility for himself, under the fixed bids of other buyers and the maximum equilibrium mechanism prefixed by the market maker. The best response strategy varies depending on the circumstances of the market and has its nature place in the analysis of dynamics of strategic interplays of agents.

For the special case of single item markets where the market maker has only one item to sell, maximum equilibrium corresponds to the first price auction protocol and minimum equilibrium corresponds to the second price auction protocol. The first price auction sells the item to the buyer with the highest bid at a price equal to his bid (i.e., the highest possible price so that everyone is happy with his corresponding allocation). Under a dynamic setting, buyers take turns in making their moves in response to what they are encountered in the previous round(s). In a simple setting when all buyers bid their true values initially, the winner with the highest bid will immediately shade his bid down to the smallest value so that he is still a

winner. Such best response bidding of the winner results in an outcome equivalent to the Vickrey second price auction, which charges the winner at the smallest possible price (i.e., the second highest bid) so that everyone's utility is maximized. The observation of convergence from the first price to the second price goes hand-in-hand with the seminal revenue equivalence theory in Bayesian analysis [30, 26]. Therefore, both the Bayesian model and the dynamic best response model point to the same unification result of the first price and the second price solution concepts in the single item markets.

The story is hardly ending for the process of Nash dynamics with multiple items being sold in the market, which illustrates a rich structure that a single item market does not possess. First of all, in a single item market, the best response decision of a buyer is binary, that is, he decides either to or not to compete with the highest bid of others for the item at a smallest cost. With multiple items, however, a buyer needs to make his decision in the best response over all items rather than a single item. In particular, he may desire any item in a subset — each of which brings him the same maximum utility — and lose interest for other items. The buyer's best response, therefore, is in accordance with his strategic bids for the items in *both* subsets. For example, it is possible that one increases his bid for one item but decreases for another in a best response. This opens many possibilities that would complicate the analysis; in particular, best responses of a buyer are not unique. Indeed, as Example 3.1 shows, not all best responses guarantee convergence, again because of the multiplicity in possibilities of bidding strategies.

Second, in a single item market, the submitted bids of buyers illustrate a monotone property to the price of the item. That is, when a buyer increases his bid, the price is always monotonically non-decreasing. With multiple items, while it is true that the prices of the items are still monotone with respect to the bids of the losers, counter-intuitively, the prices are no longer monotone with respect to the bids of the winners, as the following Example 1.1 shows. This non-monotonicity further requires extra efforts to analyze the multiplicity of best response strategies and their properties.

Example 1.1 (Non-monotonicity). *There are three buyers i_1, i_2, i_3 and two items j_1, j_2 with submitted bids $b_{i_1 j_1} = 2$, $b_{i_2 j_1} = 12$, $b_{i_2 j_2} = 14$ and $b_{i_3 j_2} = 5$ (the bids of all unspecified pairs are zero). For the given bid vector, $(p_{j_1}, p_{j_2}) = (3, 5)$ is the maximum equilibrium price vector, and assigning j_1 to i_2 and j_2 to i_3 are supported allocations. If a winner i_2 increases his bid of j_2 to $b'_{i_2 j_2} = 15$, then $(p'_{j_1}, p'_{j_2}) = (2, 5)$ is the maximum equilibrium price vector (with the same allocations) where the price of j_1 is decreased from 3 to 2. Hence, the maximum equilibrium prices are not monotone with respect to the submitted bids (of the winners).*

Finally, the maximum market equilibrium solution has no closed form as in the single item case, and the convergence of the best responses depends on a careful choice of the bidding strategy. Recall that a major difficulty in analyzing the best responses is that, in addition to bidding strategically for the items that bring the maximum utility, buyers may have (almost arbitrarily) different bidding strategies for those items that they are not interested in; these decisions indeed play a critical role to the convergence of the best responses. As not all best responses necessarily lead to convergence (Example 3.1), we restrict to a specific bidding strategy, called *aligned best response*, where a bid vector of a buyer is called aligned if any allocation of an item with positive bid brings him the maximum utility. The aligned bidding strategy illustrates the preference of a buyer over all items and is shown to be a best response (Lemma 3.1 and 3.2). While the aligned best response still does not have a monotone property in general (see Figure 3), the prices do exhibit a pattern of monotonicity when the bids of the buyers have already been aligned. Based on these properties, we show that the aligned best response always converges and maximizes social welfare, summarized by the following claim.

Theorem. *In the maximum competitive equilibrium mechanism game, for any initial bid vector and any ordering of the buyers, the aligned best response always converges. Further, the allocation at convergence maximizes social welfare.*

In addition to proving convergence, another important question is that which Nash equilibrium to which the best response will converge. In contrast with single item markets where Nash equilibrium is essentially unique, in multi-item markets there can be several Nash equilibria with completely irrelevant price vectors (see Appendix D); this is another remarkable difference between single item and multi-item markets. Despite

of the multiplicity of Nash equilibria, if we start with an aligned bid vector (e.g., bid truthfully), the best response always converges to one at a minimum competitive equilibrium, i.e., a VCG outcome.

Theorem. *Starting from an aligned bid vector, the aligned best response of the maximum equilibrium mechanism converges to a minimum competitive equilibrium at truthful bidding.*

1.1 Related Work and Motivation

The study of competitive equilibrium in a matching market was initiated by Shapley and Shubik in an assignment model [28]. They showed that maximum and minimum competitive equilibria always exist and gave a simple linear program to compute one. Their results were later improved to the models with general utility functions [7, 13]. Leonard [22] and Demange and Gale [8] studied strategic behaviors in the market and proved that the minimum equilibrium mechanism admits a truthful dominant strategy for every buyer. Later, Demange, Gale and Sotomayor [9] gave an ascending auction based algorithm that converges to a minimum equilibrium. Our study focuses on Nash dynamics in the matching market model; the convergence from maximum equilibrium to minimum equilibrium implies revenue equivalence, and addresses the question that why short-term revenue maximization is a poor long run strategy in a dynamic framework.

The Nash dynamics of best responses has its nature place in the analysis of interplays of strategic agents. In general, characterizing equilibria of the dynamics is difficult or intractable. There have been extensive studies in the literature for some special settings, e.g., potential games [25], congestion games [4], evolutionary games [18], concave games [11], correlated equilibrium [17], sink equilibria [14], and market equilibrium [31], to name a few. Complexity issues have also been addressed in the analysis of best responses [2, 12, 25]. Recently, Nisan et al. [27] independently considered best response dynamics in matching market and show a similar convergence result for running first price auctions for all items individually. To the best of our knowledge, our work is the first to study best response dynamics in the maximum competitive equilibrium mechanism.

Despite the motivation is mainly from theoretical curiosity, our setting does capture some realistic applications, such as eBay electronic market and sponsored search market, which have attracted a lot research efforts in recent years. The mechanism used in the sponsored search market is that of the generalized second price (GSP) auction. Because GSP is not truthful in general, a number of studies have focused on strategic considerations of advertisers. Edelman et al. [10] and Varian [29] independently showed that certain Nash equilibrium in the GSP auction derives the same revenue as the well-known truthful VCG scheme. Cary et al. [3] showed that a certain best response bidding strategy converges to the best Nash equilibrium. Recently, Leme and Tardos [21] considered other possible Nash equilibrium outputs and showed that the ratio between the worst and best Nash equilibria is upper bounded by 1.618. These results putting together illustrate a pretty complete overview of the structure of strategic behaviors in GSP. Our results are not directly about GSP but in a different way to reconfirm the revenue equivalence: While the search engine may adopt a different protocol with the goal of revenue maximization (i.e., the maximum equilibrium mechanism), with rational advertisers its overall revenue will eventually be the same as in the VCG protocol.

Another widely studied problem related to our model is that of spectrum markets. In designing the FCC spectrum auction protocol, a multiple stage bidding process, proposed by Milgrom [23], to digest coordination, optimization and withdrawal, is adopted. It is conducted in several stages to allow buyers to change their bids when the seller announces the tentative prices of the licenses for the winners. Therefore, it is created as an alternative game played between the seller and the buyers as a whole. Our best response analysis considers the dynamic aspect of the model and illustrates the convergence of the dynamic process.

Organization. We will first describe our model and maximum/minimum competitive equilibrium (mechanism) in Section 2. In Section 3, we define the aligned bidding strategy and show that it is a best response and always converges. In Section 4, we characterize Nash equilibria in the maximum equilibrium mechanism game; and based on the characterization, we show that the maximum equilibrium mechanism converges to a minimum equilibrium output. We conclude our discussions in Section 5.

2 Preliminaries

We have a market with n unit-demand buyers, where each buyer wants at most one item, and m indivisible items, where each item can be sold to at most one buyer. We will denote buyers by i and items by j throughout the paper. For every buyer i and item j , there is a *value* $v_{ij} \in [0, \infty)$, representing the maximum amount that i is willing to pay for item j . We will assume that there are m dummy buyers all with value zero for each item j , i.e., $v_{ij} = 0$. This assumption is without loss of generality, and implies that the number of items is always less than or equal to the number of buyers, i.e., $m \leq n$.

The outcome of the market is a tuple (\mathbf{p}, \mathbf{x}) , where

- $\mathbf{p} = (p_1, \dots, p_m) \geq 0$ is a *price* vector, where p_j is the price charged for item j ;
- $\mathbf{x} = (x_1, \dots, x_n)$ is an *allocation* vector, where x_i is the item that i wins. If i does not win any items, denote $x_i = \emptyset$. Note that different buyers must win different items, i.e., $x_i \neq x_{i'}$ for any $i \neq i'$ if $x_i, x_{i'} \neq \emptyset$.

Given an output (\mathbf{p}, \mathbf{x}) , let $u_i(\mathbf{p}, \mathbf{x})$ denote the *utility* that i obtains. We will assume that all buyers have quasi-linear utilities. That is, if i wins item j (i.e., $x_i = j$), his utility is $u_i(\mathbf{p}, \mathbf{x}) = v_{ij} - p_j$; if i does not win any item (i.e., $x_i = \emptyset$), his utility is $u_i(\mathbf{p}, \mathbf{x}) = 0$.

Buyers' preferences over items are according to their utilities — higher utility items are more preferable. We say that buyer i (*strictly*) *prefers* j to j' if $v_{ij} - p_j > v_{ij'} - p_{j'}$, is *indifferent* between j and j' if $v_{ij} - p_j = v_{ij'} - p_{j'}$, and *weakly prefers* j to j' if $v_{ij} - p_j \geq v_{ij'} - p_{j'}$. In particular, a utility of zero, $v_{ij} - p_j = 0$, means that i is indifferent between buying item j at price p_j and not buying anything at all; a negative utility $v_{ij} - p_j < 0$ means that the buyer strictly prefers to not buy the item at price p_j .

We consider the following solution concept in this paper.

Definition 2.1. (COMPETITIVE EQUILIBRIUM) *We say a tuple (\mathbf{p}, \mathbf{x}) is a competitive equilibrium if (i) for any item j , $p_j = 0$ if no one wins j in allocation \mathbf{x} , and (ii) for any buyer i , his utility is maximized by his allocation at the given price vector. That is,*

- *if i wins item j (i.e., $x_i = j$), then $v_{ij} - p_j \geq 0$; and for every other item j' , $v_{ij} - p_j \geq v_{ij'} - p_{j'}$;*
- *if i does not win any item (i.e., $x_i = \emptyset$), then for every item j , $v_{ij} - p_j \leq 0$. (For notational simplicity, we write $v_{i\emptyset} - p_\emptyset = 0$.)*

The first condition above is an *efficiency* condition (i.e., market clearance), which says that all unallocated items are priced at zero (or at some given reserve price). The assumption that there is a dummy buyer for each item allows us to assume, without loss of generality, that all items are allocated in an equilibrium. The second is a *fairness* condition (i.e., envy-freeness), implying that each buyer is allocated with an item that maximizes his utility at these prices. That is, if i wins item j , then i cannot obtain higher utility from any other item; and if i does not win any item, then i cannot obtain a positive utility from any item. Namely, all buyers are *happy* with their corresponding allocations at the given price vector.

For any given matching market, Shapley and Shubik [28] proved that there always is a competitive equilibrium. Actually, what they showed was much stronger — there is the unique minimum (respectively, maximum) equilibrium price vector, defined formally as follows.

Definition 2.2. (MINIMUM EQUILIBRIUM MIN-EQ AND MAXIMUM EQUILIBRIUM MAX-EQ) *A price vector \mathbf{p} is called a minimum equilibrium price vector if for any other equilibrium price vector \mathbf{q} , $p_j \leq q_j$ for every item j . An equilibrium (\mathbf{p}, \mathbf{x}) is called a minimum equilibrium (denoted by MIN-EQ) if \mathbf{p} is the minimum equilibrium price vector. The maximum equilibrium price vector and a maximum equilibrium (denoted by MAX-EQ) are defined similarly.*

For example, there are three buyers i_1, i_2, i_3 and one item; the values of buyers are $v_{i_1} = 10, v_{i_2} = 5$ and $v_{i_3} = 2$. Then $p = 5$ and $p = 10$ are the minimum and maximum equilibrium prices, respectively. Allocating the item to the first buyer i_1 , together with any price $5 \leq p \leq 10$, yields a competitive equilibrium. When

there is a single item, it can be seen that the outcome of the minimum equilibrium and the maximum equilibrium corresponds precisely to the “second price auction” and the “first price auction”, respectively.

Consider another example: There are three buyers i_1, i_2, i_3 and two items j_1, j_2 ; the values of buyers are $v_{i_1 j_1} = 10, v_{i_1 j_2} = 6, v_{i_2 j_1} = 8, v_{i_2 j_2} = 4$, and $v_{i_3 j_1} = 3, v_{i_3 j_2} = 2$. Then $(6, 2)$ is the minimum equilibrium price vector and $(8, 4)$ is the maximum equilibrium price vector. Note that the allocation vectors supported by the minimum or maximum equilibrium price vector may not be unique. In this example, we can allocate j_1 and j_2 arbitrarily to i_1 and i_2 to form an equilibrium. Indeed, as Gul and Stacchetti [16] showed, if both \mathbf{p} and \mathbf{q} are equilibrium price vectors and (\mathbf{p}, \mathbf{x}) is a competitive equilibrium, then (\mathbf{q}, \mathbf{x}) is an equilibrium as well.

2.1 Maximum Equilibrium Mechanism Game

In this paper, we will consider a maximum equilibrium as a mechanism, that is, every buyer i reports a *bid* b_{ij} for each item j (note that b_{ij} can be different from his true value v_{ij}), and given reported bids from all buyers $\mathbf{b} = (b_{ij})$, the *maximum equilibrium mechanism* (again denoted by MAX-EQ) outputs a maximum equilibrium with respect to \mathbf{b} . Let $\text{MAX-EQ}(\mathbf{b})$ denote the (maximum) equilibrium returned by the mechanism.

Let $u_i(\mathbf{b})$ denote the utility that i obtains in $\text{MAX-EQ}(\mathbf{b})$. That is, if $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, then $u_i(\mathbf{b}) = v_{ix_i} - p_{x_i}$ if $x_i \neq \emptyset$ and $u_i(\mathbf{b}) = 0$ if $x_i = \emptyset$. Note that the (true) utility of every buyer is defined according to his true valuations v_{ij} , rather than the bids b_{ij} ; and the “equilibrium” output (\mathbf{p}, \mathbf{x}) is computed in terms of the bid vector \mathbf{b} , rather than the true valuations (i.e., it is only guaranteed that $b_{ix_i} - p_{x_i} \geq b_{ij} - p_j$ for any item j). Therefore, the true utility of a buyer i might not be maximized at the corresponding allocation x_i and the given price vector \mathbf{p} . Further, the output $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$ might not even be a real equilibrium with respect to the true valuations of the buyers.

Considering competitive equilibria as mechanisms defines a multi-parameter setting and it is natural to consider strategic behaviors of the buyers. While it is a dominant strategy for every buyer to report his true valuations in the minimum equilibrium mechanism (i.e., the mechanism outputs a MIN-EQ for the given bid vector) [8], the MAX-EQ mechanism does not in general admit a dominant strategy. This is true even for the degenerated single item case — it is well-known that the first price auction is not incentive compatible. Therefore, in the MAX-EQ mechanism, buyers will behave strategically to maximize their utilities. In particular, fixing bids of other buyers, buyer i will naturally place his bid vector $(b_{ij})_j$ to maximize his own utility; such a vector is called a *best response*. The focus of the present paper is to consider convergence of best response dynamics, i.e., buyers iteratively change their bids according to best responses while all the others remain their previous bids unchanged. We say a best response dynamics *converges* if it eventually reaches a state where no buyer is willing to change his bid anymore; hence, it arrives at a Nash equilibrium.

As our focus is on the convergence of best response dynamics, we discretize the bidding space of every buyer to be a multiple of a given arbitrarily small constant $\epsilon > 0$, and assume that all v_{ij} ’s and b_{ij} ’s are multiples of ϵ . In practice, ϵ can be, e.g., one cent or one dollar (or any other unit number). This assumption is natural in the context of two-sided markets with money transfers, and implies that if a buyer increases his bid for an item, it must be by at least ϵ . In addition, we assume that the bid that every buyer submits for every item is less than or equal to his true valuation, i.e., $b_{ij} \leq v_{ij}$. This assumption is rather mild because (i) bidding higher than the true valuations carries the risk of a negative utility, and (ii) as Lemma 3.1 and 3.2 below show, for any given fixed bids of other buyers, there always is a best response strategy for every buyer to bid less than or equal to his true valuations.

2.2 Computation of Maximum Equilibrium

Shapley and Shubik [28] gave a linear program to compute a MIN-EQ; their approach can be easily transformed to compute a MAX-EQ by changing the objective to maximize the total payment. Next we give a combinatorial algorithm to iteratively increase prices to converge to a MAX-EQ. The idea of the algorithm is important to our analysis in the subsequent sections.

Definition 2.3 (Demand graph). *Given any given bid vector \mathbf{b} and price vector \mathbf{p} , its demand graph*

$G(\mathbf{b}, \mathbf{p}) = (A, B; E)$ is defined as follows: A is the set of buyers and B is the set of items, and $(i, j) \in E$ if $b_{ij} \geq p_j$ and $b_{ij} - p_j \geq b_{ij'} - p_{j'}$ for any j' .

In a demand graph $G(\mathbf{b}, \mathbf{p})$, every edge (i, j) represents that item j gives maximal utility to buyer i , presumed that his true valuations are given by \mathbf{b} . Note that demand graph is uniquely determined by the given bid vector and price vector, and is independent to any allocation vector. However, an equilibrium allocation must be selected from the set of edges in the demand graph. The following definition of alternating paths in the demand graph is crucial to our analysis. (Recall that all items are allocated out in an equilibrium.)

Definition 2.4. (max-ALTERNATING PATH) *Given any equilibrium (\mathbf{p}, \mathbf{x}) of a given bid \mathbf{b} , let $G = G(\mathbf{b}, \mathbf{p})$ be its demand graph. For any item j , a path $(j = j_1, i_1, j_2, i_2, \dots, j_\ell, i_\ell)$ in graph G is called a max-alternating path if edges are in and not in the allocation \mathbf{x} alternatively, i.e., $x_{i_k} = j_k$ for all $k = 1, \dots, \ell$. Denote by $G_j^{\max}(\mathbf{b}, \mathbf{p}, \mathbf{x})$ (or simply G_j^{\max} when the parameters are clear from the context) the subgraph of $G(\mathbf{b}, \mathbf{p})$ (containing both buyers and items including j itself) reachable from j through max-alternating paths with respect to \mathbf{x} . A max-alternating path $(j = j_1, i_1, j_2, i_2, \dots, j_\ell, i_\ell)$ in $G_j^{\max}(\mathbf{b}, \mathbf{p}, \mathbf{x})$ is called critical if $b_{i_\ell j_\ell} = p_{j_\ell}$.*

For any given bid vector \mathbf{b} , let (\mathbf{p}, \mathbf{x}) be an arbitrary equilibrium. Note that for any buyer i with $x_i \neq \emptyset$, (i, x_i) is an edge in the demand graph $G(\mathbf{b}, \mathbf{p})$. We consider the following recursive rule to increase prices: For any item j , increase prices of all items in G_j^{\max} continuously with the same amount until one of the following events occurs:

1. There is buyer $i \in G_j^{\max}$ such that $b_{ix_i} = p_{x_i}$ (note that $x_i \in G_j^{\max}$);
2. There is buyer $i \in G_j^{\max}$ and item $j' \notin G_j^{\max}$ such that i obtains maximal utility from j' as well; then we add edge (i, j') to the demand graph and update G_j^{\max} for each j .

The process continues iteratively until we cannot increase the price for any item. Denote the algorithm by ALG-MAX-EQ. We have the following result.

Theorem 2.1. *Starting from an arbitrary equilibrium (\mathbf{p}, \mathbf{x}) for the given bid vector \mathbf{b} , let \mathbf{q} be the final price vector in the above price-increment process ALG-MAX-EQ. Together with the original allocation vector \mathbf{x} , (\mathbf{q}, \mathbf{x}) is a MAX-EQ with respect to \mathbf{b} .*

The above algorithm ALG-MAX-EQ and theorem illustrate the idea of defining G_j^{\max} (and the corresponding max-alternating paths), summarized in the following corollary. (Note that if $(j = j_1, i_1, j_2, i_2, \dots, j_\ell, i_\ell)$ is a critical max-alternating path in $G_j^{\max}(\mathbf{b}, \mathbf{p}, \mathbf{x})$, then the last pair (i_ℓ, j_ℓ) where $b_{i_\ell j_\ell} = p_{j_\ell}$ is the exact reason that why we are not able to increase the price p_j further in the ALG-MAX-EQ to derive a higher equilibrium price vector.)

Corollary 2.1. *Given any bid vector \mathbf{b} and $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, for any item j , there is a critical max-alternating path in the subgraph $G_j^{\max}(\mathbf{b}, \mathbf{p}, \mathbf{x})$.*

3 Best Response Dynamics

For any given bid vector \mathbf{b} , assume that the MAX-EQ outputs (\mathbf{p}, \mathbf{x}) , i.e., $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$. If (\mathbf{p}, \mathbf{x}) is not a Nash equilibrium, there is a buyer who is able to obtain more utility by unilaterally changing his bid. Such a buyer will therefore naturally choose a vector (called a *best response*) to bid so that his utility is maximized in the MAX-EQ mechanism, given fixed bids of all other buyers.

Consider the following example: There are three buyers i_1, i_2, i_3 and two items j_1, j_2 , with $b_{i_1 j_1} = v_{i_1 j_1} = 20$, $b_{i_1 j_2} = v_{i_1 j_2} = 18$, and $b_{i_2 j_1} = b_{i_2 j_2} = b_{i_3 j_1} = b_{i_3 j_2} = 10$. Then $\mathbf{p} = (12, 10)$ is the maximum equilibrium price and i_1 obtains utility 8. Consider a scenario where i_1 changes his bid to $b'_{i_1 j_1} = b'_{i_1 j_2} = 15$; then the maximum equilibrium price vector becomes $\mathbf{p}' = (10, 10)$. Given the equilibrium price vector \mathbf{p}' , however, there are two different equilibrium allocations which give him different true utilities (where i_1 either wins j_1

with utility $v_{i_1 j_1} - p'_{j_1} = 10$ or wins j_2 with utility $v_{i_1 j_2} - p'_{j_2} = 8$). Hence, different selections of equilibrium allocations may lead to different true utilities, which in turn will certainly affect best response strategies of the buyers.

To analyze the best responses of the buyers, it is therefore necessary to specify a framework about their belief on the resulting equilibrium allocations. We will consider worst case analysis in this paper, i.e., all buyers are risk-averse and always assume the worst possible allocations when making their best responses¹. The above bid vector $b'_{i_1 j_1} = b'_{i_1 j_2} = 15$, therefore, is not a best response for i_1 since his utility in the worst allocation is only 8. In the worst-case analysis framework, the best responses of i_1 in the above example are given by $b'_{i_1 j_1} = 10 + \epsilon$ and $b'_{i_1 j_2} \leq 10$ where i_1 wins j_1 at price $10 + \epsilon$ with utility $v_{i_1 j_1} - (10 + \epsilon) = 10 - \epsilon$ (it can be seen that there is no bid vector such that i_1 can obtain a utility higher than or equal to 10 given fixed bids of the other two buyers)².

The above example further shows that best response strategies may not be unique: While i_1 always wants to bid $10 + \epsilon$ for j_1 , he can bid any value between 0 and 10 for j_2 to constitute a best response. While the bids for those items in which a buyer is not interested (i.e., j_2 in the above example) will not affect the utility that the buyer obtains after his best response bidding, they do affect the overall convergence of the best response dynamics, as the following example shows.

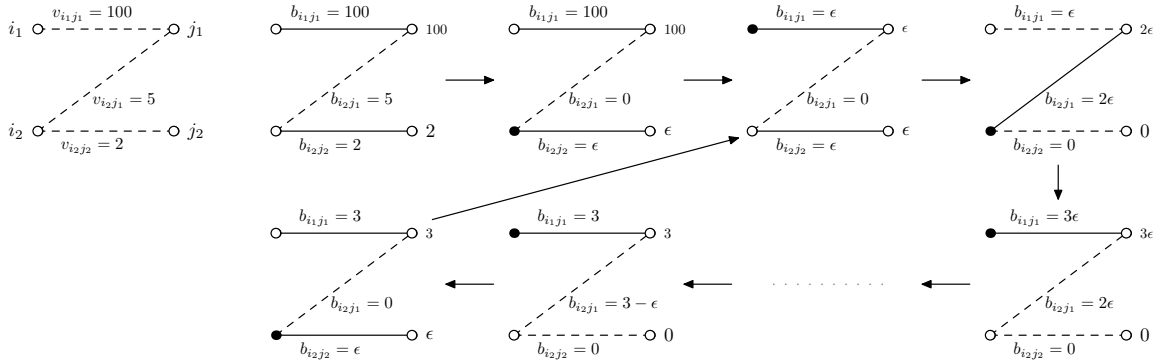


Figure 1: A best response that does not converge.

Example 3.1. (NON-CONVERGENCE OF A BEST RESPONSE) *There are two buyers i_1, i_2 and two items j_1, j_2 , where $v_{i_1 j_1} = 100$, $v_{i_1 j_2} = 0$, $v_{i_2 j_1} = 5$ and $v_{i_2 j_2} = 2$. (There still are dummy buyers; we do not describe them explicitly.) Consider the following specific best response strategy: Every buyer in the best response always bids zero to those items that he is not interested in. Since $v_{i_1 j_2} = 0$, we assume that i_1 always bids zero to j_2 . The process in Figure 1 shows that the best response dynamics does not converge. (For all examples in the paper, the black vertices on the left denote the buyers who change their bids in the best response, the solid lines denote allocations, and the numbers along with item vertices denote the maximum equilibrium prices.)*

3.1 Best Response Strategy

Given the non-convergence of a specific best response shown in Example 3.1, one would ask if there are any best response strategies that *always* converge for *any* given instance. We consider the following bidding strategy.

¹When buyers report the same bids on some items, a certain tie-breaking rule should be specified to decide an equilibrium allocation. In the literature, ties are broken either by an oracle access to the true valuations or by a given fixed order of buyers [19, 20]. In our worst-case analysis framework, we actually do not need to specify a tie-breaking rule explicitly. It essentially implies that (i) a buyer who changes his bid has the lowest priority in tie-breaking, and (ii) any buyer who bids zero cannot win the corresponding item at price zero due to the existence of dummy buyers (i.e., there is no free lunch).

²For any fixed bids of other buyers, the utility that a risk-averse buyer obtains in a best response is always within a distance of ϵ to that in a best possible allocation that a risk-seeking buyer may obtain. Thus, the worst case analysis (i.e., with risk-averse buyers) does provide a “safe” equilibrium allocation in which the corresponding utility is almost the same as the maximal.

Lemma 3.1. (BEST RESPONSE OF LOSERS) *For any given bid vector \mathbf{b} , let $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$. Given fixed bids of other buyers, consider any buyer i_0 with $x_{i_0} = \emptyset$; let $d_{i_0} = \max_j v_{i_0j} - p_j$.*

- *If $d_{i_0} \leq 0$, a best response of i_0 is to bid the same vector $(b_{i_0j})_j$.*
- *Otherwise, a best response of i_0 is given by vector $(b'_{i_0j})_j$, where $b'_{i_0j} = \max\{0, v_{i_0j} - d_{i_0} + \epsilon\}$. Further, the utility that i_0 obtains after the best response bidding is either d_{i_0} or $d_{i_0} - \epsilon$.³*

Lemma 3.2. (BEST RESPONSE OF WINNERS) *For any given bid vector \mathbf{b} , let $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$. Given fixed bids of other buyers, consider any buyer i_0 with $x_{i_0} \neq \emptyset$. Denote by $\mathbf{b}_{\neq i_0}$ the bid vector derived from \mathbf{b} where i_0 changes his bid to 0 for all items. Let $(\mathbf{q}, \mathbf{y}) = \text{MAX-EQ}(\mathbf{b}_{\neq i_0})$ and $d_{i_0} = \max_j v_{i_0j} - q_j$.*

- *If $d_{i_0} \leq 0$ or $u_{i_0}(\mathbf{p}, \mathbf{x}) = d_{i_0}$, a best response of i_0 is to bid the same vector $(b_{i_0j})_j$.*
- *Otherwise, a best response of i_0 is given by vector $(b'_{i_0j})_j$, where $b'_{i_0j} = \max\{0, v_{i_0j} - d_{i_0} + \epsilon\}$. Further, the utility that i_0 obtains after the best response bidding is either d_{i_0} or $d_{i_0} - \epsilon$.⁴*

The best response $(b'_{i_0j})_j$ defined in the above two lemmas has two remarkable properties: First, if $v_{i_0j} \geq v_{i_0j'}$ then $b'_{i_0j} \geq b'_{i_0j'}$; second, if $b'_{ij}, b'_{ij'} > 0$ then $v_{i_0j} - b'_{i_0j} = v_{i_0j'} - b'_{i_0j'}$. Hence, the best response $(b'_{i_0j})_j$ captures the preference of i_0 over all items. In addition, given fixed bids of other buyers, the difference $v_{i_0j} - b'_{i_0j}$ gives the maximal possible utility (up to a gap of ϵ) that i_0 is able to obtain from item j .

We say a bid vector $(b_{ij})_j$ *aligned* if for any $b_{ij}, b_{ij'} > 0$, $v_{ij} - b_{ij} = v_{ij'} - b_{ij'}$. That is, $(b_{ij})_j$ is derived from $(v_{ij})_j$ by moving the curve down in parallel capped at 0. It can be seen that the bid vectors defined in the above Lemma 3.1 and Lemma 3.2 are aligned. Hence, we will refer such bidding strategy by *aligned best response*. In the following, unless stated otherwise, all best responses are aligned according to these two lemmas. The following Figure 2 shows the convergence of Example 3.1 according to the aligned best response when both buyers bid their true values initially (as in Figure 1).

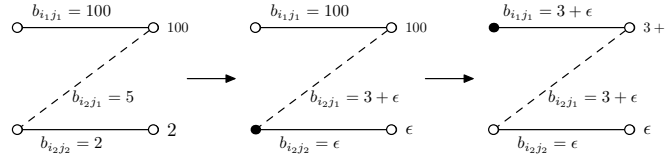


Figure 2: Convergence of Example 3.1 by the aligned best response.

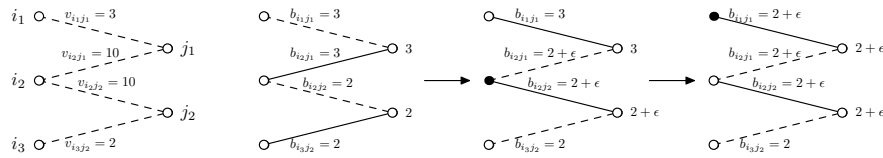


Figure 3: Non-monotonicity of the aligned best response.

³Note that if $d_{i_0} > \epsilon$, then certainly i_0 improves his utility through the best response bidding. If $d_{i_0} = \epsilon$, however, it is possible that i_0 obtains the same utility $d_{i_0} - \epsilon = 0 = u_{i_0}(\mathbf{p}, \mathbf{x})$ after the best response bidding. For such a case, $(b'_{i_0j})_j$ is still a best response and we assume that i_0 will continue to change his bid according to it. This assumption is without loss of generality as our focus is on the convergence of the best response strategy. Further, in practice, buyers usually do not have complete information about the market (e.g., bid vectors of others) and are unaware of the exact utility they will obtain if change their bids. Hence, continuing to bid the best response provides a safe strategy for the buyers to maximize their utilities.

⁴Similar to the discussions for Lemma 3.1, if $u_{i_0}(\mathbf{p}, \mathbf{x}) < d_{i_0} - \epsilon$, then i_0 strictly improves his utility after the best response bidding; if $u_{i_0}(\mathbf{p}, \mathbf{x}) = d_{i_0} - \epsilon$, it is possible that i_0 obtains the same utility $d_{i_0} - \epsilon$ after the best response bidding. Again for such a case, $(b'_{i_0j})_j$ is still a best response and we assume that i_0 will continue change his bid according to it.

3.2 Properties of the Best Response

Example 1.1 shows that maximum equilibrium prices may not be monotone with respect to the submitted bid vectors. Such non-monotonicity still occurs even for the aligned best response of the winners defined in Lemma 3.2. In the example of the Figure 3, when a winner i_2 changes his bid, the price of j_2 is increased from 2 to $2 + \epsilon$ in the maximum equilibrium. Hence, the maximum equilibrium prices may not be monotonically decreasing with respect to the aligned best responses of the winners.

However, as the following claim shows, we do have monotonicity for the maximum equilibrium prices given a certain condition. This property is crucial in the analysis of convergence.

Lemma 3.3. *When a loser makes a best response bidding (by Lemma 3.1), the maximum equilibrium prices will not decrease. On the other hand, when a winner, who has already made at least one aligned best response bidding, makes a best response bidding (by Lemma 3.2), the maximum equilibrium prices will not increase.*

We have the following corollary.

Corollary 3.1. *Given a bid vector \mathbf{b} and $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, any loser i_0 is willing to make a best response bidding only if there is an item j such that $v_{i_0 j} > p_j$. For any winner i_0 who has already made a best response bidding, the following claims hold:*

- i_0 is willing to make a best response bidding \mathbf{b}' only if the price of item x_{i_0} will decrease in $\text{MAX-EQ}(\mathbf{b}_{\neq i_0})$ (defined in Lemma 3.2).
- Let $(\mathbf{q}, \mathbf{y}) = \text{MAX-EQ}(\mathbf{b}_{\neq i_0})$ and $(\mathbf{p}', \mathbf{x}') = \text{MAX-EQ}(\mathbf{b}')$, then $(\mathbf{p}', \mathbf{x})$ is a MAX-EQ for \mathbf{b}' as well. Hence, we can assume without loss of generality that when i_0 places a best response bidding, the allocation remains the same.

3.3 Convergence of the Best Response Dynamics

In this subsection we will show that the aligned best response defined in Lemma 3.1 and 3.2 always converges. In the rest of this subsection we will assume that all buyers have already made a best response bidding; this assumption is without loss of generality since our goal is to prove convergence. Hence, the results established in the last subsection can be applied directly.

Proposition 3.1. *Given a bid vector \mathbf{b} and $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, assume that a winner i_0 first makes a best response bidding.*

- If next another winner i'_0 makes a best response bidding, then i_0 will not change his bid in the best response.
- If next a loser i'_0 makes a best response bidding, then as long as i_0 is still a winner after i'_0 's bid, he will not change his bid in the best response.

Similar to Proposition 3.1, we have the following claim for the best response of losers.

Proposition 3.2. *Given a bid vector \mathbf{b} and $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, assume that a loser i_0 first makes a best response bidding and becomes a winner.*

- If next another winner i'_0 makes a best response bidding, then i_0 will not change his bid in the best response.
- If next another loser i'_0 makes a best response bidding, then as long as i_0 is still a winner after i'_0 's bid, he will not change his bid in the best response.

The above two propositions imply the following corollary.

Corollary 3.2. *Given a bid vector \mathbf{b} and $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$,*

- if a winner i_0 makes a best response bidding, then unless i_0 becomes a loser (due to the best responses of others), he will not change his bid in the best response;
- if a loser i_0 makes a best response bidding and becomes a winner, then unless i_0 becomes a loser again (due to the best responses of others), he will not change his bid in the best response.

We are now ready to prove our main theorem.

Theorem 3.1. *In the maximum competitive equilibrium mechanism game, for any given initial bid vector and any ordering of the buyers, the aligned best response defined in Lemma 3.1 and 3.2 always converges.*

Proof. By Corollary 3.2, after all buyers have already made a best response bidding, only losers are willing to make best responses to become a winner and winners are willing to make best responses only if they become a loser. By Lemma 3.3, the prices will be monotonically non-decreasing when losers make best responses. Hence, the process eventually terminates. \square

The above theorem only says that the aligned best response is guaranteed to converge in finite steps; indeed it may take a polynomial of $\max_{ij} v_{ij}/\epsilon$ steps to converge. This is true even for the simplest setting with one item and two buyers of the same value $v_1 = v_2$: The two buyers with initial bids zero keep increasing their bids by 2ϵ one by one to beat each other until the price reaches to $v_1 = v_2$. However, in most applications like advertising markets, the value of $\max_{ij} v_{ij}$ is rather small. Further, to guarantee efficient convergence, we may set ϵ to be sufficiently large so that the rally between the winners and losers is fast (e.g., in most ascending auctions, the minimum increment of a bid is scaled according to the expected value of $\max_{ij} v_{ij}$).

4 Characterization of Nash Dynamics: From MAX-EQ to MIN-EQ

Theorem 3.1 in the previous section shows that the aligned best response always converges for any given initial bid vector. However, it does not answer the questions of at which condition(s) the dynamics of the best response stops (i.e., reaches to a stable state), and how the stable state looks like. We will answer these questions in this section.

Theorem 4.1. *For any Nash equilibrium bid vector \mathbf{b} of the MAX-EQ mechanism, let $(\mathbf{p}, \mathbf{x}) = \text{MIN-EQ}(\mathbf{b})$ and $(\mathbf{q}, \mathbf{y}) = \text{MAX-EQ}(\mathbf{b})$. Then for any item j , we have either $p_j = q_j$ or $p_j + \epsilon = q_j$. That is, the MAX-EQ and MIN-EQ price vectors differ by at most ϵ , i.e., $\text{MAX-EQ}(\mathbf{b}) \approx \text{MIN-EQ}(\mathbf{b})$.*

The above characterizations apply to *all* Nash equilibria in the MAX-EQ mechanism, and gives a necessary condition that when a bid vector forms a Nash equilibrium. Note that the other direction of the claim does not hold. For example, there are one item and two buyers with the same true value $v_1 = v_2 = 10$. In the bid vector \mathbf{b} with $b_1 = b_2 = 5$, we have $\text{MAX-EQ}(\mathbf{b}) = \text{MIN-EQ}(\mathbf{b})$. However, \mathbf{b} is not a Nash equilibrium as the loser in the $\text{MAX-EQ}(\mathbf{b})$ can increase his bid to win the item.

In general, there can be multiple Nash equilibria for a given market. To compare different Nash equilibria, we use a universal benchmark — the solution given by the truthful MIN-EQ mechanism, i.e., $(\mathbf{p}^*, \mathbf{x}^*) = \text{MIN-EQ}(\mathbf{v})$, where \mathbf{v} is the true valuation vector. This is equivalent to the outcome of the second price auction in single item markets and the VCG mechanism for the general multi-item markets. In Appendix D, we give two examples to show that, for a given Nash equilibrium \mathbf{b} with $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, there could be no fixed relation between the price vectors \mathbf{p} and \mathbf{p}^* , i.e., it can be either $\mathbf{p} \gg \mathbf{p}^*$ or $\mathbf{p} \ll \mathbf{p}^*$. This is a remarkable difference between single item and multi-item markets — in the former we always have $\mathbf{p} \approx \mathbf{p}^*$ (i.e., $\|\mathbf{p} - \mathbf{p}^*\| \leq \epsilon$).

However, if we initially start with an aligned bid vector for all buyers (e.g., bid truthfully), then there is a strong connection between the MAX-EQ price vector at a converged Nash equilibrium and $(\mathbf{p}^*, \mathbf{x}^*)$: The two price vectors are “almost” identical, up to a gap of ϵ . We summarize our results in the following theorem.

Theorem 4.2. *For any Nash equilibrium bid vector \mathbf{b} converged from the aligned best response, starting from an aligned bid vector for all buyers, let $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$ and $(\mathbf{p}^*, \mathbf{x}^*) = \text{MIN-EQ}(\mathbf{v})$, where $\mathbf{v} = (v_{ij})$ is the true valuation vector. Then we have*

- either $p_j = p_j^*$ or $p_j = p_j^* + \epsilon$ for all items;
- $\text{MAX-EQ}(\mathbf{b}) \approx \text{MIN-EQ}(\mathbf{b}) \approx \text{MIN-EQ}(\mathbf{v})$.

We comment that if we start with an *arbitrary* bid vector, as long as all buyers bid at least one aligned best response in the dynamics, the convergence result from MAX-EQ to MIN-EQ still holds. In particular, this includes the case when all buyers bid very low at the beginning. In addition, for any given bid vector \mathbf{b} , if the MAX-EQ mechanism outputs $\text{MAX-EQ}(\mathbf{b}) = \text{MIN-EQ}(\mathbf{v})$, it is not necessary that \mathbf{b} is a Nash equilibrium. For example, it is possible that the bids of losers are quite small and therefore winners still want to decrease their bids to pay less.

Finally, we characterize allocations at all Nash equilibria. We say an allocation \mathbf{x} *efficient* if $\sum_i v_{ix_i}$ is maximized, i.e., social welfare is maximized. It is well-known that all allocations in competitive equilibria are efficient at truthful bidding. This can be easily generalized to say that for any given bid vector \mathbf{b} , any equilibrium allocation \mathbf{x} maximizes $\sum_i b_{ix_i}$. We have the following result, which says that any Nash equilibrium converged from the aligned best response actually maximizes the real social welfare with respect to \mathbf{v} .

Theorem 4.3. *For any Nash equilibrium bid vector \mathbf{b} of the MAX-EQ mechanism converged from the aligned best response (starting from an arbitrary bid vector), let $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$. Then \mathbf{x} is an efficient allocation that maximizes social welfare.*

We note that the above claim does not hold for general Nash equilibria. For example, buyers i_1 and i_2 are interested in item j_1 at values $v_{i_1 j_1} = v$ and $v_{i_2 j_1} = v + \epsilon$. Then allocating the item to i_1 at price v is a Nash equilibrium for the bid vector $b_{i_1 j_1} = b_{i_2 j_1} = v$ (since the second buyer i_2 still obtains a non-positive utility even if he can win the item at a higher bid). This allocation does not maximize social welfare, and the example can be easily generalized to arbitrary number of items with large deficiency. In the aligned best response bidding, however, i_2 will continue to bid $v + \epsilon$ to win the item by Lemma 3.1, which leads to an efficient allocation.

5 Concluding Remarks

In this paper, we analyze the dynamics of best responses of the maximum competitive equilibrium mechanism in a matching market. While a best response strategy may not necessarily converge, we show that a specific bidding strategy, aligned best response, always converges to a Nash equilibrium. The outcome at such a Nash equilibrium is actually a minimum equilibrium given truthful bidding if we start with an aligned bid vector. In other words, our results show that maximum equilibrium converges to minimum equilibrium, which is a reminiscence of the convergence of dynamic bidding from first price auction to second price auction in a single item market.

In our discussions, we assume that all buyers have complete information for the market, including, say, submitted bid vectors of others, market prices and their corresponding allocation. We note that the first one, submitted bids of other buyers, is not necessary in the dynamics: The best response of losers defined in Lemma 3.1 applies automatically, and the best response of winners defined in Lemma 3.2 can be implemented by a two-step strategy without requiring extra information in the market.

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A Proof of Theorem 2.1

Lemma A.1. *Let (\mathbf{p}, \mathbf{x}) and $(\mathbf{p}', \mathbf{x}')$ be any two equilibria for the given bid vector \mathbf{b} . Let $T = \{j \mid p_j < p'_j\}$ and $S = \{i \mid x'_i \in T\}$ be the subset of buyers who win items in T in the equilibrium $(\mathbf{p}', \mathbf{x}')$. Then all buyers in S win all items in T in the equilibrium (\mathbf{p}, \mathbf{x}) as well. In particular, this implies that if j is allocated to a dummy buyer in (\mathbf{p}, \mathbf{x}) (i.e., essentially no one wins j), then its price is zero in all equilibria.*

Proof. Note that since $p'_j > p_j \geq 0$ for any $j \in T$, all items in T must be sold out in equilibrium $(\mathbf{p}', \mathbf{x}')$; hence, $|S| = |T|$ and all buyers in S win all items in T in $(\mathbf{p}', \mathbf{x}')$. Consider any buyer $i \in S$, let $j' = x'_i \in T$. For any item $j \notin T$, since $p_j \geq p'_j$, we have

$$b_{ij'} - p_{j'} > b_{ij'} - p'_{j'} \geq b_{ij} - p'_j \geq b_{ij} - p_j$$

where the second inequality follows from the fact that $(\mathbf{x}', \mathbf{p}')$ is an equilibrium. Hence, buyer i always strictly prefers item j' to all other items not in T at price vector \mathbf{p} , which implies that his allocation $x_i \in T$. \square

Proof of Theorem 2.1. By the rule of increasing prices, since the initial (\mathbf{p}, \mathbf{x}) is an equilibrium, all buyers are always satisfied with their respective allocations in the course of the algorithm. Further, it can be seen that any item that is allocated to a dummy buyer (i.e., it has initial price $p_j = 0$) will never have its price increased. Hence, the final output (\mathbf{q}, \mathbf{x}) is an equilibrium as well.

Assume that $(\mathbf{x}^*, \mathbf{q}^*)$ is a MAX-EQ for the given bid vector \mathbf{b} ; we know that $q_j \leq q_j^*$ for all items. Let $T = \{j \mid q_j < q_j^*\}$ and $S = \{i \mid x_i^* \in T\}$ be the subset of buyers who win items in T in the MAX-EQ equilibrium. By Lemma A.1, we know that all buyers in S win all items in T in equilibrium (\mathbf{q}, \mathbf{x}) as well. Assume that $T \neq \emptyset$; and consider any $j_1 \in T$ and the subgraph $G_{j_1}^{\max}(\mathbf{b}, \mathbf{q}, \mathbf{x})$ reachable from j_1 in the demand graph of the final output (\mathbf{q}, \mathbf{x}) .

We claim that all items in $G_{j_1}^{\max}$ are in T . Otherwise, consider an item $j_\ell \in G_{j_1}^{\max} \setminus T$ and the max-alternating path $(j_1, i_1, j_2, i_2, \dots, i_{\ell-1}, j_\ell)$ defining j_ℓ to be in $G_{j_1}^{\max}$. Assume without loss of generality that j_ℓ is the first item on the path which is not in T , i.e., $j_{\ell-1} \in T$ and $j_\ell \notin T$. since $j_{\ell-1} = x_{i_{\ell-1}} \in T$, by Lemma A.1, we have $j^* \triangleq x_{i_{\ell-1}}^* \in T$. Hence,

$$\begin{aligned} b_{i_{\ell-1}j_\ell} - q_{j_\ell}^* &\geq b_{i_{\ell-1}j_\ell} - q_{j_\ell} \\ &= b_{i_{\ell-1}j_{\ell-1}} - q_{j_{\ell-1}} \\ &\geq b_{i_{\ell-1}j^*} - q_{j^*} \\ &> b_{i_{\ell-1}j^*} - q_{j^*}^* \end{aligned}$$

which contradicts to the fact that $i_{\ell-1}$ obtains his utility-maximized item in the MAX-EQ.

Since all items in $G_{j_1}^{\max}$ have their prices increased, again by Lemma A.1, all items in $G_{j_1}^{\max}$ are sold out. Therefore, at the end of the algorithm when reaching to (\mathbf{q}, \mathbf{x}) , we should still be able to increase prices for items in $G_{j_1}^{\max}$, which is a contradiction. That is, $T = \emptyset$ and (\mathbf{q}, \mathbf{x}) is a MAX-EQ. \square

B Proofs in Section 3

B.1 Proof of Lemma 3.1

Proof. Given fixed bids of other buyers, consider any bid vector $(b_{i_0j}^*)_j$ of buyer i_0 . Denote the resulting bid vector by \mathbf{b}^* , and let $(\mathbf{p}^*, \mathbf{x}^*) = \text{MAX-EQ}(\mathbf{b}^*)$. A basic observation is that no matter what bid that i_0 submits, everyone other than i_0 is still happy with the original equilibrium (\mathbf{p}, \mathbf{x}) . Hence, if $b_{i_0j}^* \leq p_j$ for any j , then (\mathbf{p}, \mathbf{x}) is still a MAX-EQ for \mathbf{b}^* . Otherwise, let $j_1 \in \arg \max_j \{b_{i_0j}^* - p_j\}$. Consider subgraph $G_{j_1}^{\max}(\mathbf{b}, \mathbf{p}, \mathbf{x})$ and its critical max-alternating path $(j_1, i_1, j_2, i_2, \dots, j_\ell, i_\ell)$ with respect to \mathbf{x} , where $x_{i_k} = j_k$ for $k = 1, \dots, \ell - 1$ and the pair i_ℓ and j_ℓ is the reason that p_{j_1} cannot be increased in (\mathbf{p}, \mathbf{x}) by the algorithm (i.e., $b_{i_\ell j_\ell} = p_{j_\ell}$). Consider reallocating each j_k to i_{k-1} for $k = 2, \dots, \ell$. By the definition of max-alternating path, we know that all these buyers are still happy with their new allocations. Further, we reallocate j_1 to i_0 ; then i_0 obtains his utility-maximized item at price \mathbf{p} under bid vector \mathbf{b}^* . This new allocation, together with the price vector \mathbf{p} , constitutes an equilibrium. In both cases, \mathbf{p} is an equilibrium price vector under bid vector \mathbf{b}^* . Hence, the price of every item in $\text{MAX-EQ}(\mathbf{b}^*)$ is larger than or equal to p_j , i.e., $p_j^* \geq p_j$.

We next analyze the best response of i_0 defined in the statement of the claim. If $v_{i_0j} \leq p_j$ for any j , then i_0 cannot get a positive utility from any item at price vector \mathbf{p} , as well as \mathbf{p}^* . Hence, bidding the original (losing) price vector is a best response strategy. It suffices to consider there is an item j such that $v_{i_0j} > p_j$ and the best response strategy $(b'_{i_0j})_j$ described in the second part of the statement (denoted by \mathbf{b}').

Let $T = \{j \mid v_{i_0j} - p_j = d_{i_0}\}$; by the assumption, $T \neq \emptyset$. It can be seen that for any $j \in T$, $b'_{ij} - p_j = \epsilon$; and any $j \notin T$, $b'_{ij} - p_j \leq v_{i_0j} - d_{i_0} + \epsilon - p_j < d_{i_0} - d_{i_0} + \epsilon = \epsilon$, i.e., $b'_{ij} - p_j \leq 0$. That is, given bid vector \mathbf{b}' and price vector \mathbf{p} , i_0 always desires those items in T . Consider any item $j_0 \in T$, by the same reassignment argument described above, we can reallocate j_0 to i_0 , as well as a few other reallocations through a critical max-alternating path, to derive an equilibrium $(\mathbf{p}, \mathbf{x}')$, where \mathbf{x}' is the corresponding new allocation. Note that $(\mathbf{p}, \mathbf{x}')$ may not be a MAX-EQ. Consider subgraph $G_{j_0}^{\max}(\mathbf{b}', \mathbf{p}, \mathbf{x}')$; we ask whether p_{j_0} can be increased further by the algorithm ALG-MAX-EQ to get a MAX-EQ.

If the answer is ‘no’, then $(\mathbf{p}, \mathbf{x}')$ is indeed a MAX-EQ under bid vector \mathbf{b}' (it can be shown that the price of any other item cannot be increased as well), and the utility that i_0 obtains satisfies

$$\begin{aligned} u_{i_0}(\mathbf{p}, \mathbf{x}') &= v_{i_0j_0} - p_{j_0} = d_{i_0} \\ &= \max_j v_{i_0j} - p_j \geq \max_j v_{i_0j} - p_j^* \geq u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) \end{aligned}$$

If the answer is ‘yes’, then we can increase prices of all items in $G_{j_0}^{\max}(\mathbf{b}', \mathbf{p}, \mathbf{x}')$ by ϵ , which gives an equilibrium $(\mathbf{p}', \mathbf{x}')$, where $p'_j = p_j + \epsilon$ if $j \in G_{j_0}^{\max}(\mathbf{b}', \mathbf{p}, \mathbf{x}')$ and $p'_j = p_j$ otherwise. Further, $(\mathbf{p}', \mathbf{x}')$ is a MAX-EQ under bid vector \mathbf{b}' since the price of j_0 is tight with the bid of i_0 , i.e., $b'_{i_0j_0} = p'_{j_0}$ (again, prices of other items cannot be increased). Hence, the utility that i_0 obtains is

$$u_{i_0}(\mathbf{p}', \mathbf{x}') = v_{i_0j_0} - p'_{j_0} = v_{i_0j_0} - p_{j_0} - \epsilon = d_{i_0} - \epsilon$$

Next we consider the utility that i_0 obtains in the equilibrium $(\mathbf{p}^*, \mathbf{x}^*)$ with bid vector \mathbf{b}^* . Note that all other buyers bid the same values in \mathbf{b}, \mathbf{b}' and \mathbf{b}^* . Let $j_1 \triangleq x_{i_0}^*$; then

$$u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) = v_{i_0j_1} - p_{j_1}^* \leq v_{i_0j_1} - p_{j_1} \leq d_{i_0}$$

If one of the above inequalities is strict, then $u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) < d_{i_0}$. That is, $u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) \leq d_{i_0} - \epsilon = u_{i_0}(\mathbf{p}', \mathbf{x}')$, which implies that \mathbf{b}' is a best response strategy. It remains to consider the case when all inequalities are tight, i.e., $u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) = d_{i_0}$ and $p_{j_1}^* = p_{j_1}$; in this case, we have $j_1 \in T$. Consider the following two cases regarding the relation between j_0 and j_1 .

- If $j_0 = j_1$, consider the subgraph $G_{j_1}^{\max}(\mathbf{b}^*, \mathbf{p}^*, \mathbf{x}^*)$. Since the price of j_1 cannot be increased, there is a critical max-alternating path inside $P = (j_1, i_1 = i_0, j_2, i_2, \dots, j_\ell, i_\ell)$ where $x_{i_k}^* = j_k$ and $b_{i_\ell j_\ell}^* = p_{j_\ell}^*$. Since

$$b_{i_0j_2}^* - p_{j_2}^* = b_{i_0j_1}^* - p_{j_1}^* = b_{i_0j_1}^* - p_{j_1}$$

we know that $j_2 \in T$ and $p_{j_2}^* = p_{j_2}$ (otherwise, $v_{i_0 j_2} - p_{j_2}^* < d_{i_0}$, then in the worst allocation the utility of i_0 is less than d_{i_0}). We claim that all items in P have $p_j^* = p_j$. Otherwise, consider the first item j_k where $p_{j_k}^* > p_{j_k}$. Note that $k \geq 3$; then we have

$$\begin{aligned} b'_{i_{k-1} j_{k-1}} - p_{j_{k-1}} &= b_{i_{k-1} j_{k-1}}^* - p_{j_{k-1}}^* \\ &= b_{i_{k-1} j_k}^* - p_{j_k}^* \\ &< b_{i_{k-1} j_k}^* - p_{j_k} \\ &= b'_{i_{k-1} j_k} - p_{j_k} \end{aligned}$$

Hence, $x'_{i_{k-1}} \in T^* \triangleq \{j \mid p_j^* > p_j\}$ and $x_{i_{k-1}}^* = j_{k-1} \notin T^*$. This implies that there must be a buyer i such that $x'_i \notin T^*$ and $x_i^* \in T^*$, which is impossible since i does not get his utility-maximized item in $(\mathbf{p}, \mathbf{x}')$. Therefore, essentially P defines a critical max-alternating path in $G_{j_0}^{\max}(\mathbf{b}', \mathbf{p}, \mathbf{x}')$.

- If $j_0 \neq j_1$, starting from $P' = (j_0, i_0, j_1)$, we expand the path P' through the following rule: if the current last edge is (i_k, j_{k+1}) , expand (j_{k+1}, i_{k+1}) if $x'_{i_{k+1}} = j_{k+1}$; if the current last edge is (j_k, i_k) , expand (i_k, j_{k+1}) if $x_{i_k}^* = j_{k+1}$. The process stops when there is no more item or buyer to expand; denote the final path by $P' = (j_0, i_0, j_1, i'_1, j'_2, i'_2, \dots, j_\ell, i'_\ell)$. Note that edges in P' are in \mathbf{x}' and \mathbf{x}^* alternatively. Further, since i'_1 wins j_1 in \mathbf{x}' and j'_2 in \mathbf{x}^* , plus the fact that $p_{j_1}^* = p_{j_1}$, we have $p_{j'_2}^* = p_{j'_2}$; then since i'_2 wins j'_2 in \mathbf{x}' and j'_3 in \mathbf{x}^* , we have $p_{j'_3}^* = p_{j'_3}$; the argument inductively implies that for every item $j \in P'$, $p_j^* = p_j$. At the end of path P' , i'_ℓ does not win any item in \mathbf{x}^* , which implies that $b'_{i'_\ell j'_\ell} = p_{j'_\ell}$. If we consider path P' in $G_{j_0}^{\max}(\mathbf{b}', \mathbf{p}, \mathbf{x}')$, the above arguments show that it is actually a critical max-alternating path.

Hence, in both cases we cannot increase price p_{j_0} in the equilibrium $(\mathbf{p}, \mathbf{x}')$, which contradicts to our assumption that the answer is ‘yes’. \square

B.2 Proof of Lemma 3.2

Proof. Given fixed bids of other buyers, consider any bid vector $(b_{i_0 j}^*)_j$ of buyer i_0 . Denote the resulting bid vector by \mathbf{b}^* , and let $(\mathbf{p}^*, \mathbf{x}^*) = \text{MAX-EQ}(\mathbf{b}^*)$. Consider the two equilibria (\mathbf{q}, \mathbf{y}) and $(\mathbf{p}^*, \mathbf{x}^*)$ in the following virtual scenario: i_0 first bids 0 and loses in the MAX-EQ (\mathbf{q}, \mathbf{y}) and then bids according to \mathbf{b}^* yielding a new MAX-EQ $(\mathbf{p}^*, \mathbf{x}^*)$. By a similar argument as the proof of the above lemma, we know that $q_j \leq p_j^*$ for all items. In particular, the argument applies to the case when $\mathbf{b}^* = \mathbf{b}$, hence $q_j \leq p_j$.

Since the bid of any buyer is always less than or equal to his true value, we have

$$u_{i_0}(\mathbf{p}, \mathbf{x}) = v_{i_0 x_{i_0}} - p_{x_{i_0}} \geq b_{i_0 x_{i_0}} - p_{x_{i_0}} \geq 0$$

If $u_{i_0}(\mathbf{p}, \mathbf{x}) = d_{i_0}$, then certainly i_0 cannot obtain more utility when bidding \mathbf{b}^* since

$$u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) = v_{i_0 x_{i_0}^*} - p_{x_{i_0}^*}^* \leq v_{i_0 x_{i_0}^*} - q_{x_{i_0}^*} \leq d_{i_0} = u_{i_0}(\mathbf{p}, \mathbf{x})$$

If $d_{i_0} \leq 0$, then

$$v_{i_0 x_{i_0}} - p_{x_{i_0}} \leq v_{i_0 x_{i_0}} - q_{x_{i_0}} \leq 0$$

and

$$u_{i_0}(\mathbf{p}^*, \mathbf{x}^*) = v_{i_0 x_{i_0}^*} - p_{x_{i_0}^*}^* \leq v_{i_0 x_{i_0}^*} - q_{x_{i_0}^*} \leq 0$$

Hence, $u_{i_0}(\mathbf{p}, \mathbf{x}) = 0 \geq u_{i_0}(\mathbf{p}^*, \mathbf{x}^*)$, which implies that bidding the original vector $(b_{i_0 j})_j$ is a best response strategy.

It remains to consider $d_{i_0} > u_{i_0}(\mathbf{p}, \mathbf{x}) \geq 0$ and analyze the best response $(b'_{i_0 j})_j$, denote by \mathbf{b}' , defined in the statement of the claim. Consider a virtual scenario where i_0 first bids 0 and loses in the equilibrium (\mathbf{q}, \mathbf{y}) . By the above Lemma 3.1 for loser’s best response, we know that \mathbf{b}' is a best response strategy for i_0 in the virtual scenario and $u_{i_0}(\mathbf{b}') \geq d_{i_0} - \epsilon \geq u_{i_0}(\mathbf{p}, \mathbf{x})$. Since bidding according to \mathbf{b}^* is a specific strategy for i_0 , we have $u_{i_0}(\mathbf{b}') \geq u_{i_0}(\mathbf{p}^*, \mathbf{x}^*)$. These two inequalities together imply that bidding according to \mathbf{b}' is a best response strategy for i_0 . \square

B.3 Proof of Lemma 3.3

Proof. The first part of the claim follows directly from the proof of Lemma 3.1, thus we will only prove the second part. Consider bid vector \mathbf{b} and equilibrium $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, bid vector $\mathbf{b}_{\neq i_0}$ (derived from \mathbf{b} where a winner i_0 bids 0 for all items) and equilibrium $(\mathbf{q}, \mathbf{y}) = \text{MAX-EQ}(\mathbf{b}_{\neq i_0})$, and best response \mathbf{b}' , all defined in the statement of Lemma 3.2. If i_0 does not change his bid, then the maximum equilibrium prices remain the same. Hence, in the following we assume that i_0 changes his bid according to \mathbf{b}' and analyze the relation between \mathbf{p} and \mathbf{p}' , where $(\mathbf{p}', \mathbf{x}') = \text{MAX-EQ}(\mathbf{b}')$. What we need to show is that $\mathbf{p}' \leq \mathbf{p}$.

By the proof of Lemma 3.2, we know that $q_j \leq p_j$ for all items. If $\mathbf{q} = \mathbf{p}$, i.e., $q_j = p_j$ for all items, let $j^* = \arg \max_j v_{ij} - q_j$. Note that $d_{i_0} = v_{i_0 j^*} - q_{j^*} > 0$. Since (\mathbf{p}, \mathbf{x}) is an equilibrium with respect to \mathbf{b} , we have $b_{i_0 x_{i_0}} - p_{x_{i_0}} \geq b_{i_0 j^*} - p_{j^*}$ and $b_{i_0 x_{i_0}} \neq 0$. Since i_0 has already made a best response bidding (either as a loser or a winner), his bids for different items are aligned, i.e., $v_{i_0 x_{i_0}} - b_{i_0 x_{i_0}} \geq v_{i_0 j^*} - b_{i_0 j^*}$. Thus,

$$u_{i_0}(\mathbf{p}, \mathbf{x}) = v_{i_0 x_{i_0}} - p_{x_{i_0}} \geq v_{i_0 j^*} - p_{j^*} = v_{i_0 j^*} - q_{j^*} = d_{i_0}$$

Hence, i_0 already obtains his maximally possible utility and will not change his bid. In the following, we assume $\mathbf{q} \leq \mathbf{p}$ and there is an item such that $q_j < p_j$.

Let $T = \{j \mid q_j < p_j\}$ and $S = \{i \mid x_i \in T\}$ be the set of buyers who win items in T in (\mathbf{p}, \mathbf{x}) . We claim that $i_0 \in S$. Otherwise, all buyers in S win all items in T in (\mathbf{q}, \mathbf{y}) as well with positive utilities and they strictly prefer their corresponding allocations to those items that are not in T . Hence, we can increase the prices of all items in T by ϵ to derive another equilibrium; this contradicts to the fact that (\mathbf{q}, \mathbf{y}) is a MAX-EQ. Therefore, there is exactly one buyer $i^* \notin S$ who wins an item in T in (\mathbf{q}, \mathbf{y}) given bid vector $\mathbf{b}_{\neq i_0}$, i.e., $y_{i^*} \in T$. Next when i_0 changes his bid vector according to \mathbf{b}' , by the proof of Lemma 3.1, the new allocation vector \mathbf{x}' , together with the given price vector \mathbf{q} , constitutes an equilibrium. Let $j_0 = x'_{i_0}$; since

$$\begin{aligned} v_{i_0 j_0} - q_{j_0} &= \max_j v_{i_0 j} - q_j = d_{i_0} \\ &> u_{i_0}(\mathbf{p}, \mathbf{x}) \\ &= v_{i_0 x_{i_0}} - p_{x_{i_0}} \\ &\geq v_{i_0 j_0} - p_{j_0} \end{aligned}$$

we must have $j_0 \in T$. That is, when i_0 “joins the market again” from $\mathbf{b}_{\neq i_0}$ to \mathbf{b}' , he grabs an item in T “again”. Since every buyer in $S \setminus \{i_0\}$ obtains a positive utility for his corresponding allocation in T in (\mathbf{q}, \mathbf{y}) and strictly prefers it to those items that are not in T , he has to win an item in T in $(\mathbf{q}, \mathbf{x}')$. Hence, the only buyer who is kicked out of winning an item in T is i^* . That is, buyers in S “again” win all items in T . Finally, without loss of generality, we can assume that in \mathbf{x}' all items that are not in T have the same allocations as \mathbf{x} ; this still keeps an equilibrium and will not affect the computation of the maximum equilibrium price vector. Our argument above can be summarized as below:

bid vector	equilibrium	winners for items in T
\mathbf{b}	MAX-EQ (\mathbf{p}, \mathbf{x})	$S \setminus \{i_0\} \cup \{i_0\}$
$\mathbf{b}_{\neq i_0}$	MAX-EQ (\mathbf{q}, \mathbf{y})	$S \setminus \{i_0\} \cup \{i^*\}$
\mathbf{b}'	$(\mathbf{q}, \mathbf{x}')$	$S \setminus \{i_0\} \cup \{i_0\}$
	MAX-EQ $(\mathbf{p}', \mathbf{x}')$	

Finally we consider running the algorithm ALG-MAX-EQ on equilibrium $(\mathbf{q}, \mathbf{x}')$ to derive the maximum equilibrium $(\mathbf{p}', \mathbf{x}')$. By the proof of Lemma 3.1, we have either $p'_j = q_j$ or $p'_j = q_j + \epsilon$. To the end of proving $p'_j \leq p_j$, it remains to show that it is impossible that $p'_j = q_j + \epsilon = p_j + \epsilon$ for all items; assume otherwise that there is such an item j_ℓ (this implies that $j_\ell \notin T$). Then again by the proof of Lemma 3.1, the price of j_ℓ is increased (from \mathbf{q} to \mathbf{p}') through a max-alternating path $P = (j_0, i_0, j_1, i_1, \dots, j_{\ell-1}, i_{\ell-1}, j_\ell)$ in the subgraph $G_{j_0}^{\text{max}}(\mathbf{b}', \mathbf{q}, \mathbf{x}')$. That is, for $k = 0, 1, \dots, \ell - 1$, $x'_{i_k} = j_k$ and $v_{i_k j_k} - q_{j_k} = v_{i_k j_{k+1}} - q_{j_{k+1}}$. Consider subgraph $G_{j_\ell}^{\text{max}}(\mathbf{b}, \mathbf{p}, \mathbf{x})$, since $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$, we cannot increase p_{j_ℓ} and there is a critical max-alternating path P' in $G_{j_\ell}^{\text{max}}(\mathbf{b}, \mathbf{p}, \mathbf{x})$. Further, it can be seen that all items in P' are not in T (since every buyer in P' strictly prefers the corresponding allocation in \mathbf{x} to those items in T at price vector \mathbf{p}).

Since all items that are not in T have the same allocations in \mathbf{x} and \mathbf{x}' , we can expand path P through P' , which gives a critical max-alternating path in $G_{j_0}^{\max}(\mathbf{b}', \mathbf{q}, \mathbf{x}')$. Therefore, we cannot increase the price of any item in $P \cup P'$, including j_ℓ . This contradicts to our assumption that $p'_{j_\ell} = q_{j_\ell} + \epsilon$; hence the lemma follows. \square

B.4 Proof of Corollary 3.1

Proof. The claims for best responses follow directly from Lemma 3.1 and 3.2, and the proof of Lemma 3.3. For the last claim, note that if (\mathbf{q}, \mathbf{x}) is an equilibrium for \mathbf{b}' , then by the algorithm ALG-MAX-EQ to increase prices to derive the maximum equilibrium price vector \mathbf{p}' , $(\mathbf{p}', \mathbf{x})$ is a MAX-EQ for \mathbf{b}' . Recall in the proof of Lemma 3.3, $(\mathbf{q}, \mathbf{x}')$ is an equilibrium of \mathbf{b}' and all items that are not in T have the same allocations as \mathbf{x} , where $T = \{j \mid q_j < p_j\}$. Thus, it remains to show that all items in T have the same allocations in \mathbf{x} and \mathbf{x}' . Since all items in T are allocated to the same subset of buyers S in both \mathbf{x} and \mathbf{x}' , for any buyer $i \in S$, we have

$$\begin{aligned} u_i(\mathbf{p}, \mathbf{x}) &= b_{ix_i} - p_{x_i} \geq b_{ix'_i} - p_{x'_i} \\ u_i(\mathbf{q}, \mathbf{x}') &= b_{ix'_i} - q_{x'_i} \geq b_{ix_i} - q_{x_i}, \text{ if } i \neq i_0 \end{aligned}$$

and

$$\begin{aligned} u_{i_0}(\mathbf{q}, \mathbf{x}') &= b'_{i_0x'_{i_0}} - q_{x'_{i_0}} \geq b'_{i_0x_{i_0}} - q_{x_{i_0}} \\ \implies b_{i_0x'_{i_0}} - q_{x'_{i_0}} &\geq b_{i_0x_{i_0}} - q_{x_{i_0}} \end{aligned}$$

(The last inequality is because of the following argument: Assume that $b'_{i_0x'_{i_0}} = b_{i_0x'_{i_0}} - \delta$. Since i_0 has already made a best response bidding, his bid vector over different items has already been aligned. By the best responses defined in Lemma 3.1 and 3.2, we have $v_{i_0x'_{i_0}} - b_{i_0x'_{i_0}} \leq v_{i_0x_{i_0}} - b'_{i_0x'_{i_0}}$. Hence $\delta \geq 0$ and $b'_{i_0x_{i_0}} = \max\{0, b_{i_0x_{i_0}} - \delta\} \geq b_{i_0x_{i_0}} - \delta$. Therefore, $b_{i_0x'_{i_0}} - q_{x'_{i_0}} = \delta + b'_{i_0x'_{i_0}} - q_{x'_{i_0}} \geq \delta + b'_{i_0x_{i_0}} - q_{x_{i_0}} \geq b_{i_0x_{i_0}} - q_{x_{i_0}}$.) If items in T are allocated in different ways in \mathbf{x} and \mathbf{x}' , then there are $T' \subseteq T$ and $S' \subseteq S$ such that items in T' are allocated to buyers in S' in \mathbf{x} and \mathbf{x}' which forms an augmenting path. Adding the above two inequalities (regarding $u_i(\mathbf{p}, \mathbf{x})$ and $u_i(\mathbf{q}, \mathbf{x}')$) for all buyers in S' gives $\sum_{j \in T'} p_j + q_j \geq \sum_{j \in T'} p_j + q_j$. Hence, all inequalities are tight, which implies that every buyer gets the same utility from x_i and x'_i . Therefore, we can change the allocation of items in T' according to \mathbf{x} , which still gives an equilibrium. \square

B.5 Proof of Proposition 3.1

Proof. Assume that $(\mathbf{q}, \mathbf{y}) = \text{MAX-EQ}(\mathbf{b}_{\neq i_0})$ and $(\mathbf{p}', \mathbf{x}) = \text{MAX-EQ}(\mathbf{b}')$, where \mathbf{b}' is the resulting bid vector after i_0 makes his best response bidding (by Corollary 3.1, we can assume that the equilibrium allocation is the same for \mathbf{b} and \mathbf{b}'). Further, by the proof of Corollary 3.1, we know that (\mathbf{q}, \mathbf{x}) is an equilibrium for \mathbf{b}' . In the two allocations \mathbf{x} and \mathbf{y} , consider the following alternating path:

$$P = (i_0, j_0 = x_{i_0}, i_1, j_1, \dots, i_\ell, j_\ell, i_{\ell+1})$$

where i_k wins j_k in \mathbf{x} and i_{k+1} wins j_k in \mathbf{y} , and $x_{i_{\ell+1}} = \emptyset$ (this implies, in particular, $b_{i_{\ell+1}j_\ell} = q_{j_\ell}$). Further, we have

$$b_{i_{k+1}j_k} - q_{j_k} = b_{i_{k+1}j_{k+1}} - q_{j_{k+1}}, \text{ for } k = 0, \dots, \ell$$

We will analyze the relation between \mathbf{q} and the price vector after i'_0 makes his best response bidding to show the desired result.

- First consider the next best response is made by another winner i'_0 . After i'_0 makes his best response bidding denote the resulting bid vector by \mathbf{b}'' , by Corollary 3.1, we know that all winners have the same allocations \mathbf{x} . Further, the above set of equations still holds for all buyers in P for \mathbf{b}'' . This is

because, if i'_0 is one of them, say $i_{k+1} = i'_0$, then both $b_{i_{k+1}j_k}$ and $b_{i_{k+1}j_{k+1}}$ are reduced by the same amount to derive $b''_{i_{k+1}j_k}$ and $b''_{i_{k+1}j_{k+1}}$; so we still have $b''_{i_{k+1}j_k} - q_{j_k} = b''_{i_{k+1}j_{k+1}} - q_{j_{k+1}}$. Hence, the price of any item j in path P , as well as those that give the maximal utility to i_0 , cannot be smaller than q_j in $\text{MAX-EQ}(\mathbf{b}'')$ (otherwise, $i_{\ell+1}$ has to be a winner).

Therefore, when i_0 bids zero for all items after i'_0 makes his best response bidding, the price of item j_0 cannot be smaller than q_{j_0} in a $\text{MAX-EQ}(\mathbf{b}''_{\not\equiv i_0})$ as reallocating items according to P where $i_{\ell+1}$ becomes a winner gives an equilibrium allocation. By Corollary 3.1, which says that every winner obtains the same item after his best response bidding, we know that the best response of i_0 is to bid the same vector, i.e., do not change his bid.

- Next consider the next best response is made by a loser i'_0 ; let \mathbf{b}'' be the resulting bid vector. By the proof of Lemma 3.1, let $(\mathbf{p}', \mathbf{x}')$ be an equilibrium of \mathbf{b}'' , where \mathbf{p}' is the maximum equilibrium price of \mathbf{b}'' defined above and \mathbf{x}' is derived from \mathbf{x} (the allocation before i'_0 's bid) through an alternating path:

$$P' = (i'_0, j'_1, i'_1, \dots, j'_r, i'_r)$$

where i'_k wins j'_k in \mathbf{x} and i'_k wins j'_{k+1} in \mathbf{x}' , and i'_r does not win in \mathbf{x}' . (Note that to derive a MAX-EQ for \mathbf{b}'' , we still need to verify if the price of j'_1 can be increased in the subgraph $G_{j'_1}^{\text{max}}(\mathbf{b}'', \mathbf{p}', \mathbf{x}')$.)

Assume that i_0 is still a winner after i'_0 's bid; note that the item that i_0 wins in \mathbf{x}' can be either the one defined above according to P' or the same item $j_0 = x_{i_0}$. Next we consider the setting when i_0 bids zero for all items, i.e., $\mathbf{b}''_{\not\equiv i_0}$, for the two possibilities respectively.

If it is the former, i.e., $i'_k = i_0, j'_k = j_0$ and $x'_{i_0} = j'_{k+1}$, then the price of j'_k and j'_{k+1} cannot be smaller than $p'_{j'_k}$ (which is at least $q_{j'_k}$) and $p'_{j'_{k+1}}$ (which is at least $q_{j'_{k+1}}$) respectively in a $\text{MAX-EQ}(\mathbf{b}''_{\not\equiv i_0})$ as we are able to reallocate items j'_{k+1}, \dots, j'_r back to i'_{k+1}, \dots, i'_r , respectively, where j'_r becomes a winner. Since d_{i_0} defined in Lemma 3.2 will not increase, the best response of i_0 is to bid the same vector, i.e., do not change his bid.

If it is the latter, i.e., i_0 wins the same item j_0 in \mathbf{x} and \mathbf{x}' , we claim that the price of j_0 cannot be smaller than q_{j_0} in a $\text{MAX-EQ}(\mathbf{b}''_{\not\equiv i_0})$. This is because: (i) We can reallocate items according to P such that $i_{\ell+1}$ becomes a winner. (ii) If any item in P' appears in P or $i'_0 = i_{\ell+1}$ (i.e., i'_0 is the last buyer on path P), similar to the above arguments, we can reallocate items according P and P' such that i'_r becomes a winner. (Note that for the last case $i'_0 = i_{\ell+1}$, since $b_{i'_0 j_\ell} = b_{i_{\ell+1} j_\ell} = q_{j_\ell}$ and the bids of i'_0 have already been aligned, j_ℓ will give the maximal utility for i_0 when its price is q_{j_ℓ} .)

□

B.6 Proof of Proposition 3.2

Proof. An equivalent way to consider the best response of i_0 is that he first bids 0 for all items (which yields the same $\text{MAX-EQ}(\mathbf{p}, \mathbf{x})$), then bids according to the best response of winners defined in Lemma 3.2. Then we can apply the same argument as Proposition 3.1 to get the desired result. □

B.7 Proof of Corollary 3.2

Proof. The proof of the claim is by induction on the process that buyers make the best response bidding. For every such best response bidding, we use the same proof in the above Proposition 3.1 and 3.2 to get the desired result. □

C Proofs in Section 4

To prove our results, we need the following definition, which is similar to max-alternating path.

Definition C.1 (min-alternating path). *Given any equilibrium (\mathbf{p}, \mathbf{x}) of a given bid \mathbf{b} , let $G = G(\mathbf{b}, \mathbf{p})$ be its demand graph. For any item j , a path $(j = j_1, i_1, j_2, i_2, \dots, j_\ell, i_\ell)$ in G is called a min-alternating path if edges are not in and in the allocation \mathbf{x} alternatively, i.e., $x_{i_k} = j_{k+1}$ for all possible k . Denote by $G_j^{\min}(\mathbf{b}, \mathbf{p}, \mathbf{x})$ (or simply G_j^{\min} when the parameters are clear from the context) the subgraph of $G(\mathbf{b}, \mathbf{p})$ (containing both buyers and items including j itself) reachable from j through min-alternating paths with respect to \mathbf{x} . A min-alternating path $(j = j_1, i_1, j_2, i_2, \dots, j_\ell, i_\ell)$ in $G_j^{\min}(\mathbf{b}, \mathbf{p}, \mathbf{x})$ is called critical if $x_{i_\ell} = \emptyset$ and $b_{i_\ell j_\ell} = p_{j_\ell}$.*

Note that the major difference between max and min-alternating paths is that in the former, edges in the path are *in* and *not in* the allocation \mathbf{x} alternatively; whereas in the latter, edges in the path are *not in* and *in* \mathbf{x} alternatively. Similar to Corollary 2.1, we have the following claim. (The last pair j_ℓ and (i_ℓ) in a critical min-alternating path is the exact reason that why the price p_j cannot be decreased further, since otherwise i_ℓ will have to be a winner and items are over-demanded.)

Corollary C.1. *Given any bid vector \mathbf{b} and $(\mathbf{p}, \mathbf{x}) = \text{MIN-EQ}(\mathbf{b})$, for any item j , there is a critical min-alternating path in $G_j^{\min}(\mathbf{b}, \mathbf{p}, \mathbf{x})$.*

C.1 Proof of Theorem 4.1

Proof. By the definition of (\mathbf{p}, \mathbf{x}) and (\mathbf{q}, \mathbf{y}) , we have $\mathbf{p} \leq \mathbf{q}$. Let $T = \{j \mid p_j + \epsilon < q_j\}$ and $S = \{i \mid y_i \in T\}$, i.e., S is the subset of buyers that win items in T in the MAX-EQ (\mathbf{q}, \mathbf{y}) . Then for any $j \notin T$, either $p_j = q_j$ or $p_j + \epsilon = q_j$. Assume that $T \neq \emptyset$; similar to the proof of Lemma A.1, we know that all buyers in S still win items in T in the MIN-EQ (\mathbf{p}, \mathbf{x}) .

Consider any $i_0 \in S$ and let $j_0 = y_{i_0} \in T$. Consider a bid vector $(b'_{i_0 j_0})_j$ where $b'_{i_0 j_0} = q_{j_0} - \epsilon$ and $b'_{i_0 j} = 0$ for any $j \neq j_0$; denote the resulting bid vector by \mathbf{b}' (where the bids of all other buyers remain the same). Consider a tuple $(\mathbf{q}', \mathbf{y}')$, where $q'_j = q_j - \epsilon$ if $j \in T$ and $q'_j = q_j$ if $j \notin T$, and $y'_i = y_i$ if $i \in S$ and $y'_i = x_i$ if $i \notin S$. It can be seen that $(\mathbf{q}', \mathbf{y}')$ is an equilibrium for \mathbf{b}' . Note that $b'_{i_0 j_0} = q_{j_0} - \epsilon = q'_{j_0}$; thus, i_0 cannot obtain j_0 if its price q'_{j_0} is increased any further.

Let $(\mathbf{q}^*, \mathbf{y}^*) = \text{MAX-EQ}(\mathbf{b}')$; note that $\mathbf{q}' \leq \mathbf{q}^*$. We claim that i_0 still wins j_0 at price $q_{j_0}^* = q'_{j_0}$ in $(\mathbf{q}^*, \mathbf{y}^*)$. (This fact implies that i_0 obtains more utility from j_0 by paying ϵ less when bidding $(b'_{i_0 j_0})_j$; thus MAX-EQ (\mathbf{q}, \mathbf{y}) is not a Nash equilibrium.) Assume otherwise, then i_0 does not win any item in $(\mathbf{q}^*, \mathbf{y}^*)$ since $b'_{i_0 j} = 0$ for any $j \neq j_0$. Because $q_{j_0}^* \geq q'_{j_0} > p_{j_0} \geq 0$, there must be a (non-dummy) buyer winning j_0 : assume that i_1 wins $j_0 = y'_{i_0}$, i_2 wins $j_1 \triangleq y'_{i_1}$, i_3 wins $j_2 \triangleq y'_{i_2}$, ..., i_k wins $j_{k-1} \triangleq y'_{i_{k-1}}$ in $(\mathbf{q}^*, \mathbf{y}^*)$, and i_k is the first buyer in the chain that is not in S (such buyer must exist since i_0 is not a winner). Let $j_k \triangleq y'_{i_k} = x_{i_k}$ (note that $j_k \notin T$); then we have $q_{j_k}^* \geq q'_{j_k} = q_{j_k} \geq p_{j_k}$ and at least one of the two inequalities is strict (since i_k wins another item j_{k-1} at a higher price compared to (\mathbf{p}, \mathbf{x})). This implies that there must be another buyer i_{k+1} winning j_k in $(\mathbf{q}^*, \mathbf{y}^*)$. In the process all dummy buyers will not be introduced to win any item; hence, the same argument continues and will not stop, which contradicts to the fact that buyers and items are finite. \square

C.2 Proof of Theorem 4.2

The claim follows from the following two claims.

Proposition C.1. *For any item j , $p_j \geq p_j^*$.*

Proof. Assume otherwise that there is an item j_1 such that $p_{j_1} < p_{j_1}^*$. Assume that i_1 wins j_1 in \mathbf{x}^* and j_2 in \mathbf{x} , i.e., $x_{i_1}^* = j_1$ and $x_{i_1} = j_2$. Since $(\mathbf{p}^*, \mathbf{x}^*)$ is an equilibrium for the true valuation vector \mathbf{v} , we have $v_{i_1 j_1} - p_{j_1}^* \geq v_{i_1 j_2} - p_{j_2}^*$. Hence, if $p_{j_2} \geq p_{j_2}^*$, then

$$v_{i_1 j_1} - p_{j_1} > v_{i_1 j_1} - p_{j_1}^* \geq v_{i_1 j_2} - p_{j_2}^* \geq v_{i_1 j_2} - p_{j_2}$$

That is, $u_{i_1}(\mathbf{p}, \mathbf{x}) = v_{i_1 j_2} - p_{j_2} < v_{i_1 j_1} - p_{j_1} \leq d_{i_1}$ (defined in Lemma 3.2). If i_1 has yet made any best response bidding, then he should bid his best response according to Lemma 3.2. If he has already made

best response bidding, then his bid vector has already been aligned and the above inequality implies that $b_{i_1 j_1} - p_{j_1} > b_{i_1 j_2} - p_{j_2}$, which contradicts to the fact that (\mathbf{p}, \mathbf{x}) is an equilibrium for the bid vector \mathbf{b} . Hence, we must have $p_{j_2} < p_{j_2}^*$.

Consider the subgraph G given by the exclusive-OR operation of the two allocations \mathbf{x} and \mathbf{x}^* . For any alternating path $(i'_1, j'_1, \dots, i'_\ell, j'_\ell, i'_{\ell+1})$ where i'_k wins j'_k in $(\mathbf{p}^*, \mathbf{x}^*)$ and i'_{k+1} wins j'_k in (\mathbf{p}, \mathbf{x}) , if $p_{j'_k} < p_{j'_k}^*$ for any k , then by the above argument and considering buyer i'_k , we have $p_{j'_{k-1}} < p_{j'_{k-1}}^*$. Applying the same argument recursively yields $p_{j'_1} < p_{j'_1}^*$. Since $0 \leq v_{i'_1 j'_1} - p_{j'_1}^* < v_{i'_1 j'_1} - p_{j'_1}$ and i'_1 does not win any item in (\mathbf{p}, \mathbf{x}) , by Corollary 3.1 buyer i'_1 should continue to make a best response bidding, which contradicts to the fact that (\mathbf{p}, \mathbf{x}) is a Nash equilibrium. Hence, for any alternating path in G , all items have $p_j \geq p_j^*$. For any alternating cycle in G , if there is an item with $p_j < p_j^*$, by applying the above argument for j_1 recursively, all items in the cycle have $p_j < p_j^*$.

Continue to consider the above item j_1 with $p_{j_1} < p_{j_1}^*$. Since $(\mathbf{p}^*, \mathbf{x}^*) = \text{MIN-EQ}(\mathbf{v})$, by Corollary C.1 and abusing notations, there is a critical min-alternating path $P = (j_1, i'_2, j'_2, \dots, i'_{r-1}, j'_{r-1}, i'_r)$ in $G_{j_1}^{\min}(\mathbf{v}, \mathbf{p}^*, \mathbf{x}^*)$, where $x_{i'_k}^* = j'_k$ for $k = 2, \dots, r-1$, i'_r does not win any item and $v_{i'_r j'_{r-1}} = p_{j'_{r-1}}^*$. Consider buyer i'_2 ; by the definition of $G_{j_1}^{\min}$, we have

$$v_{i'_2 j_1} - p_{j_1} > v_{i'_2 j_1} - p_{j_1}^* = v_{i'_2 j'_2} - p_{j'_2}^* \geq v_{i'_2 x_{i'_2}^*} - p_{x_{i'_2}^*}^*$$

By the same above argument, we have $p_{x_{i'_2}^*} < p_{x_{i'_2}^*}^*$. Then considering the alternating cycle containing $j'_2 = x_{i'_2}^*, i'_2, x_{i'_2}$, we have $p_{j'_2} < p_{j'_2}^*$. The same argument applies to all items in P recursively; hence, we have $p_j < p_j^*$ for all items in the path P . This implies, in particular, that i'_r is a winner in (\mathbf{p}, \mathbf{x}) since $v_{i'_r j'_{r-1}} = p_{j'_{r-1}}^* > p_{j'_{r-1}}$ and by the best response characterized in Corollary 3.1.

Consider the item that i'_r wins in (\mathbf{p}, \mathbf{x}) ; let $j'_r = x_{i'_r}$. Since any buyer in an alternating cycle of G win in both (\mathbf{p}, \mathbf{x}) and $(\mathbf{p}^*, \mathbf{x}^*)$, i'_r and j'_r must be at an endpoint of an alternating path of G . Further, by the above argument regarding the prices of items in alternating paths, we have $p_{j'_r} \geq p_{j'_r}^*$. Therefore,

$$v_{i'_r j'_{r-1}} - p_{j'_{r-1}} > 0 \geq v_{i'_r j'_r} - p_{j'_r}^* \geq v_{i'_r j'_r} - p_{j'_r}$$

That is, $u_{i'_r}(\mathbf{p}, \mathbf{x}) = v_{i'_r j'_r} - p_{j'_r} < v_{i'_r j'_{r-1}} - p_{j'_{r-1}} \leq d_{i'_r}$ (defined in Lemma 3.2). We can apply the same argument above for i_1 to get a similar contradiction.

Hence, the claim follows. \square

Proposition C.2. *For every item j , if $p_j > p_j^* + \epsilon$ then the corresponding winner can obtain more utility by bidding his best response.*

Proof. From the above claim, we have $\mathbf{p}^* \leq \mathbf{p}$. Let $T_1 = \{j \mid p_j^* + \epsilon < p_j\}$ and $T_2 = \{j \mid p_j^* + \epsilon = p_j\}$. and $S_1 = \{i \mid x_i \in T_1\}$ and $S_2 = \{i \mid x_i \in T_2\}$, i.e., S_1 and S_2 are the subset of buyers that win items in T_1 and T_2 in the MAX-EQ (\mathbf{p}, \mathbf{x}) , respectively. Then for any $j \notin T_1 \cup T_2$, we have $p_j^* = p_j$.

Consider any buyer i and item $j \in T_1 \cup T_2$. If i does not win any item in the MIN-EQ $(\mathbf{p}^*, \mathbf{x}^*)$, i.e., $x_i^* = \emptyset$, then $v_{ij} \leq p_j^* < p_j$, which implies that i cannot win j in (\mathbf{p}, \mathbf{x}) . If i wins an item not in $T_1 \cup T_2$ in $(\mathbf{p}^*, \mathbf{x}^*)$, then

$$v_{ix_i^*} - p_{x_i^*} = v_{ix_i^*} - p_{x_i^*}^* \geq v_{ij} - p_j^* > v_{ij} - p_j$$

This implies that if i wins j in (\mathbf{p}, \mathbf{x}) , then he should continue to make a best response bidding by Lemma 3.2, which contradicts the fact that \mathbf{b} is a Nash equilibrium. Therefore, all buyers in $S_1 \cup S_2$ win all items in $T_1 \cup T_2$ in both $(\mathbf{p}^*, \mathbf{x}^*)$ and (\mathbf{p}, \mathbf{x}) .

Assume that $T_1 \neq \emptyset$; consider any $i_0 \in S_1$ and let $j_0 = x_{i_0}^*$. Consider a bid vector $(b'_{i_0 j})_j$ where $b'_{i_0 j_0} = p_{j_0}^* + \epsilon$ and $b'_{i_0 j} = 0$ for any $j \neq j_0$; denote the resulting bid vector by \mathbf{b}' (where the bids of all other buyers remain the same). Consider a tuple $(\mathbf{p}', \mathbf{x})$, where $p'_j = p_j^* + \epsilon$ if $j \in T_1 \cup T_2$ and $p'_j = p_j^*$ if $j \notin T_1 \cup T_2$. It can be seen that $(\mathbf{p}', \mathbf{x})$ is an equilibrium for \mathbf{b}' . Note that $b'_{i_0 j_0} = p_{j_0}^* + \epsilon = p'_{j_0}$; thus, i_0 cannot obtain j_0 if its price p'_{j_0} is increased any further.

Similar to the proof of Theorem 4.1, we can show that in i_0 still wins j_0 at price p'_{j_0} in $\text{MAX-EQ}(\mathbf{b}')$. Since

$$v_{i_0 j_0} - p'_{j_0} = v_{i_0 j_0} - p_{j_0}^* - \epsilon \geq v_{i_0 x_{i_0}} - p_{x_{i_0}}^* - \epsilon > v_{i_0 x_{i_0}} - p_{x_{i_0}}$$

we know that i_0 obtains more utility when changing bids from \mathbf{b} to \mathbf{b}' , which is again a contradiction. \square

C.3 Proof of Theorem 4.3

Proof. Consider the equilibrium $(\mathbf{p}, \mathbf{x}) = \text{MAX-EQ}(\mathbf{b})$. For any loser i , by the best response rule of Lemma 3.1, we know that $v_{ij} \leq p_j$ for all items. For any winner i , we have $v_{ix_i} - p_{x_i} \geq 0$; further, for any item $j \neq x_i$, by the best response rule of Lemma 3.2, we have $u_i(\mathbf{p}, \mathbf{x}) = v_{ix_i} - p_{x_i} \geq d_i \geq v_{ij} - p_j$, where the first inequality follows from the fact that the dynamics converges, and the second inequality follows from the observation that when i bids zero, all prices will not increase.

Let \mathbf{x}^* be an efficient allocation and maximizes social welfare. Given the existence of dummy buyers, we can assume without loss of generality that all items are sold out in both \mathbf{x} and \mathbf{x}^* . Consider the exclusive-OR relation given by the two allocations; it can be seen that it contains either alternating paths of even length where both endpoints are buyers or alternating cycles.

If there is an alternating path of even length, say,

$$i_1 \xrightarrow{\mathbf{x}} j_1 \xrightarrow{\mathbf{x}^*} i_2 \xrightarrow{\mathbf{x}} j_2 \xrightarrow{\mathbf{x}^*} \dots \dots \xrightarrow{\mathbf{x}^*} i_{r-1} \xrightarrow{\mathbf{x}} j_{r-1} \xrightarrow{\mathbf{x}^*} i_r$$

By the above discussions, we have

$$\begin{aligned} v_{i_1 j_1} - p_{j_1} &\geq 0 \\ v_{i_k j_k} - p_{j_k} &\geq v_{i_k j_{k-1}} - p_{j_{k-1}}, \quad 2 \leq k \leq r-1 \\ 0 &\geq v_{i_r j_{r-1}} - p_{j_{r-1}} \end{aligned}$$

Adding these inequalities together yields

$$\sum_{(i,j) \in \mathbf{x}} v_{ij} = \sum_{1 \leq k \leq r-1} v_{i_k j_k} \geq \sum_{2 \leq k \leq r} v_{i_k j_{k-1}} = \sum_{(i,j) \in \mathbf{x}^*} v_{ij}$$

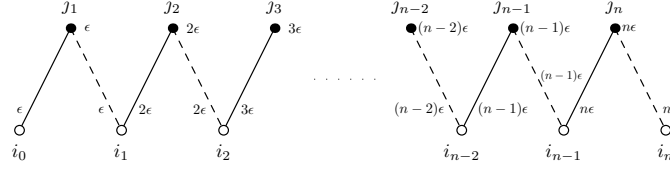
which implies that \mathbf{x} has the same maximum total valuation on such a path as \mathbf{x}^* .

The analysis is similar on alternating cycles. Hence, the allocation \mathbf{x} is efficient as well, which completes the proof. \square

D Nash Equilibria with Various Equilibrium Prices

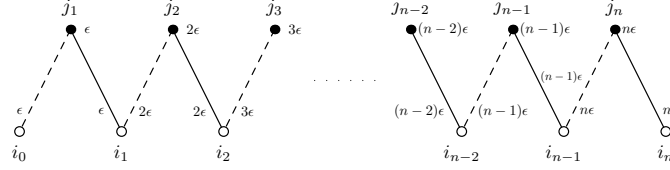
The following examples show that in general the MAX-EQ prices in a Nash equilibrium can be either (much) smaller or higher than the MIN-EQ price vector at truthful bidding. These examples are in contrast with the statement of Theorem 4.2 which says that when buyers bid aligned vectors initially, we always have $\text{MAX-EQ}(\mathbf{b}) \approx \text{MIN-EQ}(\mathbf{v})$.

Example D.1. (NASH EQUILIBRIUM WITH SMALL MAX-EQ PRICES) *There are $n+1$ buyers i_0, i_1, \dots, i_n and n items j_1, \dots, j_n . Buyer i_0 only desires j_1 with value $v_{i_0 j_1} = n\epsilon$; buyer i_k only desires j_k and j_{k+1} with value $v_{i_k j_k} = v_{i_k j_{k+1}} = n\epsilon$ for $k = 1, \dots, n-1$; and buyer i_n only desires j_n with value $v_{i_n j_n} = n\epsilon$. That is, all buyers have the same value $n\epsilon$ for the items that they desire. It can be seen that in the minimum equilibrium $(\mathbf{p}^*, \mathbf{x}^*) = \text{MIN-EQ}(\mathbf{v})$, $p_j^* = n\epsilon$ for all items. On the other hand, consider the bid vector \mathbf{b} , allocation vector \mathbf{x} (where $x_{i_k} = j_{k+1}$ and i_n does not win any item), and price vector \mathbf{p} given by the following figure: It can be seen that (\mathbf{p}, \mathbf{x}) is a MAX-EQ of \mathbf{b} . Further, \mathbf{b} is a Nash equilibrium for the MAX-EQ mechanism. (Indeed, for any buyer i_k , the maximum equilibrium price vector for bid vector $\mathbf{b}_{\neq i_k}$ where i_k bids zero for*



all items is the same as \mathbf{p} . For any buyer i_k , $1 \leq k \leq n-1$, by Lemma 3.2, the utility of i_k is maximized when bidding $b'_{i_k j_k} = b'_{i_k j_{k+1}} = (k+1)\epsilon$; in this case, however, the price of j_k will be increased to $(k+1)\epsilon$; hence, i_k will obtain the same utility as $u_{i_k}(\mathbf{p}, \mathbf{x})$. We can verify that buyers i_0 and i_n cannot obtain more utilities similarly.) Hence, the equilibrium price vector in a Nash equilibrium of the MAX-EQ mechanism can be (much) smaller than the minimum equilibrium price vector \mathbf{p}^* .

Example D.2. (NASH EQUILIBRIUM WITH LARGE MAX-EQ PRICES) There are $n+1$ buyers i_0, i_1, \dots, i_n and n items j_1, \dots, j_n . Buyer i_0 only desires j_1 with value $v_{i_0 j_1} = \epsilon$; buyer i_k only desires j_k and j_{k+1} with value $v_{i_k j_k} = v_{i_k j_{k+1}} = n\epsilon$ for $k = 1, \dots, n-1$; and buyer i_n only desires j_n with value $v_{i_n j_n} = n\epsilon$. It can be seen that in the minimum equilibrium $(\mathbf{p}^*, \mathbf{x}^*) = \text{MIN-EQ}(\mathbf{v})$, $p_j^* = \epsilon$ for all items. On the other hand, consider the bid vector \mathbf{b} , allocation vector \mathbf{x} (where $x_{i_k} = j_k$ and i_0 does not win any item), and price vector \mathbf{p} given by the following figure:



It can be seen that (\mathbf{p}, \mathbf{x}) is a MAX-EQ of \mathbf{b} . Further, \mathbf{b} is a Nash equilibrium for the MAX-EQ mechanism. (Indeed, for any buyer i_k , the maximum equilibrium price vector for bid vector $\mathbf{b}_{\neq i_k}$ where i_k bids zero for all items is the same as \mathbf{p} . Similar to the above example, we can verify that no buyer cannot obtain more utility by unilaterally changing his bid.) Hence, the equilibrium price vector in a Nash equilibrium of the MAX-EQ mechanism can be (much) larger than the minimum equilibrium price vector \mathbf{p}^* .

The above examples are not contradictory to Theorem 4.2, which says that starting from any aligned bid vector, the aligned best response dynamics converges to the MIN-EQ prices. In particular, the initial bid vectors \mathbf{b} in the above examples are not aligned; hence, the claim in Theorem 4.2 does not apply here.